Indeterminacy and Imperfect Information*

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Abstract

We study equilibrium determination in an environment where two kinds of agents have different information sets: The fully informed agents know the structure of the model and observe histories of all exogenous and endogenous variables. The less informed agents observe only a strict subset of the full information set. All types of agents form expectations rationally, but agents with limited information need to solve a dynamic signal extraction problem to gather information about the variables they do not observe. We show that for parameters values that imply a unique equilibrium under full information, the limited information rational expectations equilibrium can be indeterminate. In a simple application of our framework to a monetary policy problem we show that limited information on part of the central bank implies indeterminate outcomes even when the Taylor Principle holds.

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1 Introduction

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2 A Simple Analytical Example

2.1 Economic Framework

We consider a simple model of inflation determination. The model economy is described by a Fisher equation that links the nominal interest rate i_t to the real rate r_t via expected inflation $E_t \pi_{t+1}$, and by a monetary policy rule that has the nominal rate respond to current inflation π_t, that is, a Taylor rule.[1] We assume that the real rate is characterized by an exogenous AR(1) process with a Gaussian innovation. The equation system is thus given by:

$$i_t = r_t + E_t \pi_{t+1}, \tag{1}$$

$$i_t = \phi \pi_t, \tag{2}$$

$$r_t = \rho r_{t-1} + \varepsilon_t. \tag{3}$$

The first equation is the Fisher equation, the second is the policy rule, while the third equation describes the evolution of the real rate, where $\varepsilon_t \sim iid\ \mathcal{N}(0, \sigma_\varepsilon^2)$ and $|\rho| < 1$. All variables can be understood as deviations from their respective steady state values. ϕ is a monetary policy parameter, where we assume throughout this paper that $|\phi| > 1$.

We distinguish two agents in this economy: a representative private-sector agent whose behavior is characterized by the Fisher equation (1), and a central bank whose behavior is given by the monetary policy rule (2). We assume that the agents know the structure of the economy including the structural parameters, and that they observe the history of their respective information sets. Moreover, both agents form expectations rationally. The

[1] In the Appendix, we also consider a policy rule of the type: $i_t = r_t + \phi \pi_t$, with a time-varying intercept given by the real rate of interest. The steps towards deriving a solution are very much identical to the ones described in the main text.

central assumption of our framework, however, is that the two agents have different, but nested information sets.We describe the full information set S^t of all shocks through time t, so that for some variable x_t, $E_t x_{t+h} = E\left(x_{t+h}|S^t\right)$, for all h, and $E_t x_t = x_t$, where E_t is the rational expectations operator under full information. We define a limited information set Z^t which is nested in S^t, $Z^t \subseteq S^t$. The projections of the less informed agent for any variable x_t are denoted $x_{t|t} = E\left(x_t|Z^t\right)$ and $x_{t+h|t} = E\left(x_{t+h}|Z^t\right)$. Since Z^t is spanned by S^t we can apply the law of iterated expectations to obtain: $E\left(E(x_{t+h}|Z^t)|S^t\right) = x_{t+h|t}$. Under full information rational expectations (FIRE), both agents are assumed to know S^t. This means that they observe all variables in the model without error, that they know the history of all shocks, that they understand the structure of the economy and the solution concepts. Under limited information rational expectations (LIRE), we assume that one agent has access to the full information set S^t, while the other observes the limited information set Z^t only. For the purposes of this simple example, we assume that the private sector is fully informed whereas the central bank has limited information.

2.2 Rational Expectations Equilibria

The equation system (1) - (3) forms a linear rational expectations model that can be solved using standard methods under FIRE. When both agents have information set S^t we can find a rational expectations equilibrium (REE) as follows. Substituting the policy rule into the Fisher equation yields a relationship in inflation with driving process r_t:

$$E_t \pi_{t+1} = \phi \pi_t + r_t. \tag{4}$$

The dynamic behavior of inflation depends on the value of the policy coefficient ϕ. It is well known that the solution is unique if and only $|\phi| > 1$, in which case the determinate REE solution is $\pi_t = \frac{1}{\phi - \rho} r_t$ and $i_t = \frac{\phi}{\phi - \rho} r_t$. The properties of the exogenous process r_t carry over to inflation which is an autoregressive process.

The remainder of our paper focuses on the case when $|\phi| > 1$. Since our aim is to establish determinacy conditions under LIRE we review the implications of equilibrium indeterminacy when ϕ is inside the unit circle. In this case, the REE can be indeterminate in the sense that there are possibly infinitely many solutions that are consistent with equation (4). To describe the full set of solutions, we find it convenient to follow the approach developed by Lubik and Schorfheide (2003), which extends the Sims (2000) solution method to the case of indeterminacy. In order to find a solution to (4) we define the rational expectations forecast error $\eta_t = \pi_t - E_{t-1}\pi_t$, whereby $E_{t-1}\eta_t = 0$ by construction. This allows us to substitute out inflation expectations $E_t\pi_{t+1}$ so that we can write:

$$\pi_t = \phi\pi_{t-1} + r_{t-1} + \eta_t. \tag{5}$$

It is easily verifiable that this is a solution to the expectational difference equation (4). In this equilibrium, inflation is a stationary process with autoregressive parameter $|\phi| < 1$ and driving process r_{t-1}. What makes this equilibrium indeterminate is the fact that the solution imposes no restriction on the evolution of η_t other than that it is a martingale difference sequence with $E_{t-1}\eta_t = 0$. Consequently, there can be infinitely many solutions. We also note that the solution under indeterminacy is second-order autoregressive. Without loss of generality, we can put some structure on the solution by decomposing η_t into a fundamental component, namely the policy innovation ε_t and a non-fundamental component, the belief shock b_t, as in Farmer et al. (2015).[2] More specifically, we can write $\eta_t = \gamma_\varepsilon\varepsilon_t + \gamma_b b_t$, where $E_{t-1}b_t = 0$.[3] The unrestricted, but finite coefficients γ_ε and γ_b can be used to index specific equilibria within the set of indeterminate equilibria. In the case of FIRE the choice of these parameters is arbitrary.

We can also compute an REE under the information set Z^t of the limited information agent, namely the central bank in this example. The structure of the model, specifically the

[2] Strictly speaking, this is without loss of generality within the set of equilibria that are time-invariant and linear. There are other, non-linear equilibria that can be constructed for this linear model. See Evans and McGough (2005) for further discussion and classification.

[3] The interpretation as a belief shock in the terminology of Lubik and Schorfheide (2003) and Farmer et al. (2015) emerges when we rewrite the inflation equation in terms of expectations only. Define $\xi_t = E_t\pi_{t+1}$ and rewrite equation (4) as $\xi_t = \phi\xi_{t-1} + r_t + \phi\eta_t$. In this representation, the forecast error η_t emerges as an innovation to the conditional expectation ξ_t.

form of the equations, remains the same, but the conditioning of the expectations operator with respect to the information set changes. Specifically, we condition down the variables in the system (1) - (3) where we make use of the fact that Z^t is spanned by S^t so that we can apply the law of iterated expectations: $E\left(E(x_{t+h}|Z^t)|S^t\right) = x_{t+h|t}$. We also note that the policy rate i_t is in the central bank's information set which implies the policy rule:

$$i_t = \phi \pi_{t|t}. \tag{6}$$

Following the same steps as before, we find that:

$$\pi_{t+1|t} = \phi \pi_{t|t} + r_{t|t}, \tag{7}$$

which is a first-order difference equation in projected inflation $\pi_{t|t}$. Assuming $|\phi| > 1$, the REE solution is $\pi_{t|t} = \frac{1}{\phi-\rho} r_{t|t}$ and $i_t = \frac{\phi}{\phi-\rho} r_{t|t}$, which is isomorphic to the FIRE solution above; that is, central bank projections of the interest rate and inflation have the same relationship as the actual variables in the full information model. This solution holds under any definition of the central bank's information set as long as Z^t is spanned by S^t. What underlies this reasoning is that the central bank has less information than the private sector, but it still forms expectations rationally under its own information set, given its real rate projections $r_{t|t}$.[4]

2.3 Rational Expectations Equilibria under Asymmetric Information Sets

We now turn to the limited-information settings with rational expectations. The key element of LIRE is that there are two expectation formation processes that interact with each other. The nature of this interaction and how it affects equilibrium determination crucially depends on how the limited-information agent extracts and updates information. Our framework has

[4]This is a key difference to the framework in Lubik and Matthes (2016) who assume that the central bank engages in least-squares learning to gain information about private-sector outcomes. In our setup, the deviation from the standard rational expectations benchmark is only minor in the sense that the central bank does not observe everything that the private sector does, but is otherwise fully informed.

three building blocks. First, the relationships describing the fully-informed agent, second, those of the limited-information agent, and third, the filter used by the latter to gain additional information. In terms of the simple example, the private sector equations are given by the Fisher equation (1) and the law of motion of the real rate (3). Following Svensson and Woodford (2004), the behavior of the central bank is given by the limited information policy rule (6), where the nominal interest rate is a function of the information set Z^t. Specifically, the central bank sets the policy rate as responding to its inflation projection $\pi_{t|t}$. In addition, its behavior is constrained by its own projections, namely $\pi_{t|t} = \frac{1}{\phi-\rho} r_{t|t}$ and $i_t = \frac{\phi}{\phi-\rho} r_{t|t}$, and a projection for $r_{t|t}$. The third element is the specification of the central bank's signal extraction problem. Since the model is linear and the exogenous shocks are Gaussian, the Kalman-filter is the optimal filter in this environment. Application of the Kalman filter imposes two restrictions on the equilibrium. First, the gain in the optimal projection equation is endogenous and depends on the model's second moment. This leads to a non-trivial fixed-point problem since the model moments in turn depend on the gain. The second restriction imposes that in any equilibrium in the full model the central bank's projections have to hold; that is, rational expectations formation across all information sets has to be mutually and internally consistent. We discuss the solution of our simple framework in two steps. The central bank's projection equations depend on the information set. For purposes of exposition we distinguish between exogenous and endogenous information, whereby the former assume a noisy measurement of the real rate and the latter a noisy measurement of inflation.

2.3.1 Equilibrium with an Exogenous Information Set

Suppose that the central bank observes the real interest rate with measurement error ν_t, where $\nu_t \sim iid\ \mathcal{N}(0, \sigma_\nu^2)$. Therefore, the central bank's information set is $Z_t = r_t + \nu_t$. It is exogenous in that the real rate is an exogenous process which does not depend on other endogenous variables.[5] The Kalman projection equation for the real rate is:

$$r_{t|t} = r_{t|t-1} + \kappa_r \left(r_t - r_{t|t-1} + \nu_t \right), \tag{8}$$

where the Kalman gain κ_r is an endogenous coefficient and has to be computed separately. We now combine the private sector Fisher equation (1) with the policy rule (6):

$$\phi \pi_{t|t} = r_t + E_t \pi_{t+1}. \tag{9}$$

The evolution of inflation thus depends on two expectation formation processes: the central bank's projection of inflation $\pi_{t|t}$, and the private sector's expectation $E_t \pi_{t+1}$. It is in this sense that the two nested information sets interact. Using the formalism described above, we introduce the RE forecast error η_t and rewrite the this equation as:

$$\pi_t = \phi \pi_{t-1|t-1} - r_{t-1} + \eta_t. \tag{10}$$

We also note that the law of motion of the one-step-ahead real rate projection is $r_{t|t-1} = \rho r_{t-1|t-1}$.

We can now combine these equations into a linear RE system:

$$\begin{aligned} \pi_t &= \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \eta_t, \\ r_{t|t} &= \left(1 - \kappa_r\right) \rho r_{t-1|t-1} + \kappa_r \rho r_{t-1} + \kappa_r \varepsilon_t + \kappa_r \nu_t, \\ r_t &= \rho r_{t-1} + \varepsilon_t, \end{aligned} \tag{11}$$

where the first equation is derived from the Fisher equation with the central bank's lagged inflation projection substituted out using the restriction $\pi_{t|t} = \frac{1}{\phi - \rho} r_{t|t}$. The second equation is derived from the Kalman projection equation for the real rate and the third equation is the law of motion of the actual real rate. This is a well-specified equation system in the three

[5] However, the process of making projections of the real rate, that is, of gaining information about its true value may, and does, depend on endogenous outcomes.

unknowns inflation π_t, the exogenous real rate r_t, and the central bank projection of the real rate $r_{t|t}$. It can be solved using standard methods for linear rational expectations models that allow for indeterminacy, such as Lubik and Schorfheide (2003). What distinguishes this system from the more standard FIRE setting is that a coefficient, the gain parameter κ_r is endogenous and depends on the solution of the model; and second, the central bank's expectation $r_{t|t}$ has to be consistent with the solution of the full system it determines.

We now proceed to solve the model as follows. Following Farmer et al. (2015) we rewrite the endogenous forecast error $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t$ in terms of its stochastics components, namely the fundamnetal real rate innovation ε_t, the measurement error ν_t, and the belief shock b_t. A solution, if it exists, pins down the endogenous forecast error η_t. It is determinate if $\gamma_b = 0$ and γ_ε and γ_ν are uniquely determined. Otherwise the solution is indeterminate or an REE does not exist when no γ weights can be found that obey the restrictions imposed by on the model. In the next step, we note that the equation system is recursive and that the overall dynamic properties depend on the yet unknown value of $(1-\kappa_r)\,\rho$. In order to determine the size of this 'root' and thus the nature of the equilibrium we need to compute the gain κ_r first. For ease of notation, we find it convenient to define innovations of any variable x_t as its unexpected component relative to the limited information set Z^t: $\widetilde{x}_t = x_t - x_{t|t-1}$. We also define the projection error variance $\Sigma = var\left(\widetilde{r}_t - \widetilde{r}_{t|t}\right) = var\left(\widetilde{r}_t\right) - var\left(\widetilde{r}_{t|t}\right)$, whereby $cov(\widetilde{r}_t, \widetilde{r}_{t|t}) = var(\widetilde{r}_{t|t})$.

The steady-state Kalman gain is given by:

$$\kappa_r = \frac{cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)}{var(\widetilde{Z}_t)}, \tag{12}$$

where the tildes denote the projection innovations, e.g. $\widetilde{r}_t = r_t - r_{t|t-1}$ and $\widetilde{Z}_t = \widetilde{r}_t + \nu_t$. It can quickly be verified that $var\left(\widetilde{r}_t\right) = \rho^2\Sigma + \sigma_\varepsilon^2$ and that $var(\widetilde{Z}_t) = var\left(\widetilde{r}_t\right) + \sigma_\nu^2$. Similarly, we have $cov(\widetilde{r}_t, \widetilde{Z}_t) = var\left(\widetilde{r}_t\right)$. This leads to the following expression for the Kalman gain:

$$\kappa_r = \frac{\rho^2\Sigma + \sigma_\varepsilon^2}{\rho^2\Sigma + \sigma_\varepsilon^2 + \sigma_\nu^2}, \tag{13}$$

which for positive Σ lies within the unit circle, $0 < \kappa_r < 1$. The variance $\Sigma = var\left(\widetilde{r}_t - \widetilde{r}_{t|t}\right) = var\left(\widetilde{r}_t\right) - var\left(\widetilde{r}_{t|t}\right)$ can be computed by recognizing that $cov(\widetilde{r}_t, \widetilde{r}_{t|t}) = var(\widetilde{r}_{t|t})$ and $var\left(\widetilde{r}_{t|t}\right) = \kappa_r cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ together with the projection equation $\widetilde{r}_{t|t} = \kappa_r \widetilde{Z}_t$. Substituting these expressions into the definition of Σ results in a non-linear Riccati equation:

$$\Sigma = \frac{\rho^2 \Sigma + \sigma_\varepsilon^2}{\rho^2 \Sigma + \sigma_\varepsilon^2 + \sigma_\nu^2} \sigma_\nu^2. \tag{14}$$

The (positive) solution to this (quadratic) equation is given by:

$$\Sigma = \frac{1}{2\rho^2}\left[-\left(\sigma_\varepsilon^2 + \left(1-\rho^2\right)\sigma_\nu^2\right) + \sqrt{\left(\sigma_\varepsilon^2 + \left(1-\rho^2\right)\sigma_\nu^2\right)^2 + 4\sigma_\varepsilon^2\sigma_\nu^2\rho^2}\right]. \tag{15}$$

We can now establish that if a solution for the LIRE model with an exogeneous information set exists it is indeterminate. Since $0 < \kappa_r < 1$, it follows that $0 < \left(1-\kappa_r\right)\rho < 1$ and that the law of motion for $r_{t|t}$ in the full equation system is a stable difference equation. As the equation for actual inflation does not depend on its own lags, we can thus conclude that the equilibrium cannot be determinate. That is, the structure of the model does not impose restrictions that would uniquely pin down the endogenous forecast error η_t under the equilibrium selection criterion that the REE needs to be stationary. One such restriction would be $|\kappa_r| > 1$, which we can rule out in this case.

In the final step, we need to show that this solution is consistent with central bank projections. The condition $\pi_{t|t} = \frac{1}{\phi-\rho} r_{t|t}$ derived under the central bank's information set has to hold along any equilibrium path. It thus imposes the following restriction on innovations with respect to the central bank's information set: $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = \frac{1}{\phi-\rho} cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$, which is an optimality condition for signal extraction under the limited information set Z_t. We have already established that $cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) = \rho^2\Sigma + \sigma_\varepsilon^2$. We can write $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = cov\left(\widetilde{\pi}_t, \widetilde{r}_t\right) + cov\left(\widetilde{\pi}_t, \nu_t\right)$. Using the innovation representation of the projection equation for π_t we have:

$$\widetilde{\pi}_t = -\left(\widetilde{r}_{t-1} - \widetilde{r}_{t-1|t-1}\right) + \eta_t, \tag{16}$$

where after some substitution we find that $cov\left(\widetilde{\pi}_t, \widetilde{r}_t\right) = -\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2$. Similarly, we can find $cov\left(\widetilde{\pi}_t, \nu_t\right) = \gamma_\nu \sigma_\nu^2$. Combining all expressions then results in the following linear restriction on the weights in the forecast error $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t$:

$$\gamma_\nu = \frac{\phi}{\phi - \rho} \frac{\Sigma}{\sigma_\nu^2} + \frac{1}{\phi - \rho} \frac{\sigma_\varepsilon^2}{\sigma_\nu^2} - \frac{\sigma_\varepsilon^2}{\sigma_\nu^2} \gamma_\varepsilon. \tag{17}$$

This condition restricts the set of multiple equilibria because it restricts the γ coefficients that determine η_t to lie in a subspace of the full potential set of equilibria. This condition is absent from full information models with indeterminacy, and thus differentiates the class of LIRE models from their FIRE counterparts. At the same time, at least in the exogenous-information case, these restrictions do not affect the way belief shocks b_t ("sunspot shocks") may enter the system. We can now summarize the solution in the following

Proposition 1. *The REE in the model (1) - (3) under LIRE with the exogenous information set $Z_t = r_t + \nu_t$ is given by*

$$\begin{aligned}
\pi_t &= \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t, \\
r_{t|t} &= \left(1 - \kappa_r\right) \rho r_{t-1|t-1} + \kappa_r \rho r_{t-1} + \kappa_r \varepsilon_t + \kappa_r \nu_t, \\
r_t &= \rho r_{t-1} + \varepsilon_t,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\kappa_r &= \frac{\rho^2 \Sigma + \sigma_\varepsilon^2}{\rho^2 \Sigma + \sigma_\varepsilon^2 + \sigma_\nu^2}, \\
\Sigma &= \frac{1}{2\rho^2} \left[-\left(\sigma_\varepsilon^2 + \left(1 - \rho^2\right) \sigma_\nu^2\right) + \sqrt{\left(\sigma_\varepsilon^2 + \left(1 - \rho^2\right) \sigma_\nu^2\right)^2 + 4\sigma_\varepsilon^2 \sigma_\nu^2 \rho^2} \right], \\
-\infty &< \gamma_b < \infty, -\infty < \gamma_\varepsilon < \infty, \gamma_\nu = \frac{\phi}{\phi - \rho} \frac{\Sigma}{\sigma_\nu^2} + \left(\frac{1}{\phi - \rho} - \gamma_\varepsilon\right) \frac{\sigma_\varepsilon^2}{\sigma_\nu^2}.
\end{aligned}$$

We can also draw the following conclusions at this point. First, the limited information rational expectations equilibrium is indeterminate even though the full information counterpart has a determinate equilibrium. Indeterminacy is generic in this setting with an exogenous information set in that *any* stationary REE allows for the presence of sunspot shocks and does not uniquely determine the endogenous forecast error. From a purely mechanical perspective, the equation system (1) - (3) does not contain an unstable root despite the presence of a jump variable, namely inflation. Furthermore, it is the optimal filtering procedure employed by the central bank that introduces this stable root into the system and thus leaves the endogenous forecast error undetermined. While there is a uniquely determined mapping from the central bank's projections to endogenous outcomes, actual equilibrium outcomes, in particular the component that is orthogonal to the central bank's information set, remains indeterminate. A second observation is that under LIRE the consistency requirement for central bank projections imposes restrictions on the set of multiple equilibria. In full information solutions under indeterminacy the set of multiple equilibria is typically unrestricted, whereas optimal filtering in the LIRE counterpart restricts how the private agents coordinate on an equilibrium. From an empirical perspective, the FIRE solution results in a reduced-form representation for inflation that is first-order autoregressive. The LIRE solution on the other hand exhibits much richer dynamics. In particular, the resulting inflation process can be quite persistent when the signal-to-noise ratio is small as a large σ_ν^2 translates into a small Kalman gain. This simple example is restrictive in that the central bank only observes an exogenous process with error, whereas in practice observed variables are typically endogenous (and measured with error). In the next step we therefore asume an endogenous information set which creates additional feedback within the model.

2.3.2 Equilibrium with an Endogenous Information Set

We now assume that the central bank observes the inflation rate with measurement error ν_t such that $Z_t = \pi_t + \nu_t$. We label this an endogenous information set as the observed variable is endogenous to the solution of the model. The solution in the terms of the central bank projections conditional on the limited information set is identical to the previous case. What changes is the interdependence between the filtering problem and the equilibrium dynamics. We find it convenient to express the analysis in terms of the projection equation for the real rate to maintain comparability with the previous case. The projection equation is therefore:

$$r_{t|t} = r_{t|t-1} + \kappa_r \left(\pi_t - \pi_{t|t-1} + \nu_t \right). \tag{19}$$

From this we can derive the full equation system as before, namely:

$$\pi_t = \frac{\phi}{\phi - \rho} r_{t-1|t-1} - r_{t-1} + \eta_t, \tag{20}$$

$$r_{t|t} = (\rho + \kappa_r) r_{t-1|t-1} + \kappa_r r_{t-1} + \kappa_r \nu_t + \kappa_r \eta_t, \tag{21}$$

$$r_t = \rho r_{t-1} + \varepsilon_t. \tag{22}$$

This system looks superficially similar to the previous one with exogenous information. However, the coefficient $(\rho + \kappa_r)$ on the lagged real rate projection now depends on the endogenous dynamics of π_t via the Kalman gain κ_r in the prediction equation. Whether the equilibrium is determinate or indeterminate depends on whether the filtering implies an unstable root $|\rho + \kappa_r| > 1$; and, if not, gives rise to non-fundamental belief shocks affecting equilibrium dynamics without causing non-stationary variations. Moreover, the Kalman gain itself may no longer be unique in this setting or it may not even exist - a stark contrast from the case of exogenous information.

We proceed in solving this equation system as before. We first derive the endogenous Kalman gain and the associated forecast error variance and assess its implications for equilibrium determinacy. We then derive restrictions imposed by the central bank projections and establish consistency with the proposed equilibrium paths. The steady-state Kalman gain is given by:

$$\kappa_r = \frac{cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)}{var(\widetilde{Z}_t)}, \tag{23}$$

where $\widetilde{r}_t = r_t - r_{t|t-1}$ and $\widetilde{Z}_t = \widetilde{\pi}_t + \nu_t$. As before, we parameterize the endogenous forecast error $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t$. It can be quickly verified that $cov\left(\widetilde{r}_t, \widetilde{Z}_t\right) = -\rho\Sigma + \sigma_{\eta\varepsilon}$, where we denote $\sigma_{\eta\varepsilon}$ for notational convenience as the covariance between the endogenous forecast error and the real rate innovation. We also note that the negative sign in this expression reflects the inverse relationship between inflation and the real rate in this specification with an endogenous information set. After some algebra and making use of $\widetilde{\pi}_t = -\left(\widetilde{r}_{t-1} - \widetilde{r}_{t-1|t-1}\right) + \eta_t$, we find that $var(\widetilde{Z}_t) = var(\widetilde{\pi}_t) + var(\nu_t) + 2cov\left(\widetilde{\pi}_t, \nu_t\right)$ can be expressed as $var(\widetilde{Z}_t) = \Sigma + \sigma_\eta^2 + \sigma_\nu^2 + 2\sigma_{\eta\nu}$, where σ_η^2 and $\sigma_{\eta\nu}$ are the variance and covariance of the endogenous forecast error. We can now derive the Kalman gain as:

$$\kappa_r = \frac{-\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2}{\Sigma + \gamma_\varepsilon^2\sigma_\varepsilon^2 + \gamma_b^2\sigma_b^2 + \left(1+\gamma_\nu\right)^2\sigma_\nu^2}. \tag{24}$$

While recognizing that the forecast error variance Σ still needs to be determined as a function of the structural parameters, we can make two observations to highlight the effect of an endogenous information set. First, the gain κ_r can be negative for small enough γ_ε in contrast with the exogenous information case; that is, $\kappa_r < 0$ if $\gamma_\varepsilon < \rho\Sigma/\sigma_\varepsilon^2$. Second, it also leaves open the possibility that $|\kappa_r| > 1$, which, as can be shown, occurs for a large enough real rate innovation variance σ_ε^2. In the next step, we need to compute the projection error variance $\Sigma = var\left(\widetilde{r}_t\right) - var\left(\widetilde{r}_{t|t}\right)$. Using $var\left(\widetilde{r}_t\right) = \rho^2\Sigma + \sigma_\varepsilon^2$ and $var\left(\widetilde{r}_{t|t}\right) = \kappa_r cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ we can derive the following (quadratic) Riccatti-equation:

$$\Sigma = \rho^2\Sigma + \sigma_\varepsilon^2 - \frac{\left(-\rho\Sigma + \gamma_\varepsilon\sigma_\varepsilon^2\right)^2}{\Sigma + \sigma_\eta^2 + \sigma_\nu^2 + 2\sigma_{\eta\nu}}. \tag{25}$$

Finally, any equilibrium has to obey the restrictions imposed by the central bank projections, which is as before: $\pi_{t|t} = \frac{1}{\phi-\rho} r_{t|t}$. This implies a different covariance restriction, however, since the information set is different, namely $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = \frac{1}{\phi-\rho} cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ or alternatively $\left(\phi-\rho\right) cov\left(\widetilde{\pi}_t, \widetilde{\pi}_t + \nu_t\right) = cov\left(\widetilde{r}_t, \widetilde{\pi}_t + \nu_t\right)$. After some rearranging we can write this restriction as:

$$\gamma_\nu\left(1+\gamma_\nu\right) = -\frac{\phi}{\phi-\rho}\frac{\Sigma}{\sigma_\nu^2} - \frac{\gamma_b^2}{\phi-\rho}\frac{\sigma_b^2}{\sigma_\nu^2} + \frac{\left[1-\left(\phi-\rho\right)\gamma_\varepsilon\right]\gamma_\varepsilon}{\phi-\rho}\frac{\sigma_\varepsilon^2}{\sigma_\nu^2}. \tag{26}$$

While it looks superficially similar to the sub-space restriction derived under an exogenous information set, it is a considerably more complicated restriction, reflecting the endogeneity of the information set. In the prior case, Σ is independent of the loadings on the innovations in the forecast error decomposition, so that the restriction imposed is linear. In the current case, the loadings affect Σ and the Kalman gain. Moreover, the subspace restriction is in terms of a quadratic expression, which can imply that the solution to this equation and thus the overall equilibrium is not unique under this parameterization. Moreover, there may be no solution at all or only for a small parameter region. Before we discuss the solution in more detail, we summarize our findings in the following

Proposition 2. *Assuming $|\rho + \kappa_r| < 1$, the REE, if it exists, in the model (20) - (22) under LIRE with the endogenous information set $Z_t = \pi_t + \nu_t$ is given by*

$$\begin{aligned}
\pi_t &= \frac{\phi}{\phi-\rho} r_{t-1|t-1} - r_{t-1} + \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t, \\
r_{t|t} &= (\rho + \kappa_r) r_{t-1|t-1} - \kappa_r r_{t-1} + \kappa_r \gamma_\varepsilon \varepsilon_t + \kappa_r \gamma_b b_t + \kappa_r (1+\gamma_\nu) \nu_t, \\
r_t &= \rho r_{t-1} + \varepsilon_t,
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
\kappa_r &= \frac{-\rho\Sigma + \gamma_\varepsilon \sigma_\varepsilon^2}{\Sigma + \gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + (1+\gamma_\nu)^2 \sigma_\nu^2}, \\
\Sigma &= \frac{1}{2}\left(-\alpha \pm \sqrt{\alpha^2 - 4\beta}\right), \\
\alpha &= (1-\rho)^2 \left[\gamma_\varepsilon^2 \sigma_\varepsilon^2 + \gamma_b^2 \sigma_b^2 + (1+\gamma_\nu)^2 \sigma_\nu^2\right] - (1 + 2\rho\gamma_\varepsilon)\sigma_\varepsilon^2, \\
\beta &= -\left[\gamma_b \sigma_b^2 + (1+\gamma_\nu)^2 \sigma_\nu^2\right] \sigma_\varepsilon^2 \\
\gamma_\nu (1+\gamma_\nu) &= -\frac{\phi}{\phi-\rho}\frac{\Sigma}{\sigma_\nu^2} - \frac{\gamma_b^2}{\phi-\rho}\frac{\sigma_b^2}{\sigma_\nu^2} + \left(\frac{1}{\phi-\rho} - \gamma_\varepsilon\right)\gamma_\varepsilon \frac{\sigma_\varepsilon^2}{\sigma_\nu^2}.
\end{aligned}$$

The Proposition gives the set of solutions that apply when the equilibrium exists and is indeterminate. If $|\rho + \kappa_r| < 1$ the equation system lacks an unstable root that allows us to pin down the endogenous forecast error. We will discuss the possibility of such an alternative equilibrium in the next section. Instead the REE is indeterminate in that extraneous sunspot shocks affect the dynamics and that these dynamics are not uniquely determined on account of how the fundamental shock and the measurement error affect endogenous variables through their respective loadings γ_ε and γ_ν. Compared to the REE under FIRE, the LIRE equilibrium exhibits a considerably higher degree of serial correlation on account of the persistence imbued by the filtering process and the addition of a stable root via indeterminacy. In addition, the Proposition highlights that a solution may not exist if it violates additional restrictions. This stands in constrast with the typical notion of non-existence in RE settings, where no stationary solution can be found since the system contains too many explosive roots. What is different in the LIRE setting is that the coefficients that determine whether an equilibrium exists and whether it is unique are endogenous to the solution. While we can assess the determinacy properties of the system above for given κ_r, our framework posits that the gain, and the projection error variance, are computed based on the structural relationships given by the model. It may therefore be the case that for a given parameterization the Kalman filter does not exist, for instance, when $\Sigma < 0$. In this case the central bank's projection are inconsistent with the information structure of the economy. We also require a solution to be consistent with the central bank's projections. This imposes an additional non-linear restriction on any proposed equilibrium, given by (26) in the case of an endogenous information set.

In order to give a sense how these elements interact in determining a solution, we plot the two roots of the Riccati equation for Σ, the Kalman gain, and the subspace condition in Figure **??**. We choose a standard parameterization for illustration purposes and set $\gamma_b = 0$ for simplicity. The depicted restrictions are plotted as functions of the loading on the real rate innovation γ_ε over the range $[-10, 10]$, which in turn imply values for the loading on the measurement error γ_ν as given by (26). The hyperbola of the subspace condition reflects the quadratic on the measurement error loading γ_ν. Existence of a solution requires that the positive root of the Riccati equation and the subspace condition both hold. In this specific example, however, they never intersect or even touch. Therefore, under this parametrization no equilibrium exists since the Kalman filter does not exist in the sense that the projection equations for the variables outside of the central bank's information set are explosive and mutually inconsistent. Some experimentation for key parameters shows that the subspace condition is consistent with a positive Σ when we reduce the serial correlation of the real rate ρ, increase the policy coefficient ϕ, and also for a large measurement error variance σ_ν^2. We will present additional results and a more extensive exploration of the parameter space later on. Figure **??** also shows that the equilibrium would be indeterminate as in the case with an exogenous information set since the Kalman gain is small and negative.

2.4 Additional Results

In this section, we provide some additional insights into our framework for the simple example model that we have discussed so far. We return to these aspects in our discussion of the general framework. First, we assess the possibility that the REE equilibrium under LIRE can be determinate, depending on the existence of explosive roots in the system. If it exists, we find it convenient to label such REE our benchmark equilibrium. In the second exercise, we show that such benchmark equilibrium provides an upper variance bound for the dynamics of the model. Finally, we contrast our framework with that of Svensson and Woodford (2004) who used a minimum state variable (MSV) approach as a solution technique. We argue that they thereby miss salient aspects of this environment.

2.4.1 Determinacy in a Benchmark Equilibrium

It is well known that the determinacy properties of a linear RE model are determined by the number of unstable eigenvalues, or roots, of the underlying equation system. In a standard root-counting approach, if the number of explosive roots matches the number of forward-looking, or jump variables, the equilibrium is unique. With fewer explosive roots, the equilibrium is indeterminate and non-existent otherwise. We can apply this reasoning to our simple model under the two information sets. The respective dynamic RE equation systems are given in Propositions 1 and 2. In both cases, the system is recursive in that the dynamic properties, and thus the relevant eigenvalues, depend only on the behavior of the projection equations for the real rate. In the case of an exogenous information set, the Kalman filtering problem can be solved independently from the rest of the model since the measurement equation comprises only exogenous variables. As we have seen before, the Kalman gain $0 < \kappa_r < 1$ in this case, so that the projection equation is a stable difference equation. Given that there is one jumep variable in the system, namely π_t, thus one endogenous forecast error, the lack of an unstable root means that η_t is not pinned down and any equilibrium under this exogenous information set is indeterminate.

The case of an endogenous information set is more interesting. As before the full equation system is recursive so that we can focus on the behavior of the projection equation:

$$r_{t|t} = (\rho + \kappa_r)\, r_{t-1|t-1} + \kappa_r r_{t-1} + \kappa_r \nu_t + \kappa_r \eta_t. \tag{28}$$

We rewrite the equation slightly using the definition $r_t^* = r_t - r_{t|t}$, which is the error from the projection onto the current information set:

$$r_t^* = (\rho + \kappa_r)\, r_{t-1}^* + \varepsilon_t - \kappa_r \nu_t - \kappa_r \eta_t. \tag{29}$$

This is a first-order difference equation driven by a linear combination of shocks: the exogenous real-rate innovation ε_t, the exogenous measurement error ν_t, and the endogenous forecast error η_t. This equation is explosive if $|\rho + \kappa_r| > 1$, that is, if the Kalman gain is large enough, which is a possibility that can arise given the solution for the gain in Proposition 2.

Now suppose for the sake of argument that $|\rho + \kappa_r| > 1$. The solution to this explosive equation is $r_t^* \equiv 0$ and $\eta_t = \frac{1}{\kappa_r}\varepsilon_t - \nu_t$. It pins down the endogenous forecast error as a function of fundamentals alone. The equilibrium may thus be considered determinate as it is not affected by sunspot shocks and the system provides the necessary explosive root to match the number of jump variables. However, this is not necessarily a unique equilibrium in the sense that there is only one solution to the dynamic equation system given a set of parameters. This is because the Kalman gain κ_r is endogenous and as such there can in general be other equilibria with a different Kalman gain. Existence of this benchmark solution requires that we can find a κ_r such that $|\rho + \kappa_r| > 1$ and that the subspace condition holds. We can verify that this is in fact the case by substituting the solution into the projection equation which results in $r_t = \rho r_{t-1} + \varepsilon_t$, that is, the exogenous process for the real rate. Substituting the solution in the inflation equation yields $\pi_t = \frac{\rho}{\phi-\rho} r_{t-1} + \frac{1}{\kappa_r}\varepsilon_t - \nu_t$, which depends on the Kalman gain. In terms of the forecast error decomposition $\eta_t = \gamma_\varepsilon \varepsilon_t + \gamma_b b_t + \gamma_\nu \nu_t$ we have $\gamma_\varepsilon = 1/(\phi - \rho)$, $\gamma_\nu = -1$, and $\gamma_b = 0$. The latter follows since under determinacy in the sense of meeting the eigenvalue criterion sunspot shocks do not affect equilibrium outcomes.

This proposed equilibrium has the feature that it is what may be labelled a full revelation solution in that it implies that $r_{t|t} = r_t$; that is, the real rate projection using current information is exact. However, the equilibrium inflation rate is not revealed without error to the central bank since it depends on the measurement error ν_t. Since this solution achieves full revelation for the real rate, we hypothesize that it is consistent with the full information solution that we derived above in the following sense: $\pi_t^{FI} = \frac{\rho}{\phi-\rho} r_{t-1} + \frac{1}{\phi-\rho}\varepsilon_t$, which is void of the measurement error. Comparing the LIRE and FIRE solution we find that $\pi_t^{LI} = \pi_t^{FI} - \nu_t$, which implies $\kappa_r = \phi - \rho$. As a final step, we need to verify that this Kalman gain is consistent with the Riccati equation. Under this parametrization, the Riccati equation has a positive solution and a solution with $\Sigma = 0$, which implies full revelation of the real rate as there is no projection error. This root intersects with the subspace condition at the value $\gamma_\varepsilon = 1/(\phi - \rho)$.[6] Finally, the root of the projection equation $\rho + \kappa_r = \phi > 1$, which validates our original assumption. We summarize these finding in the following Proposition.

Proposition 3. *(Benchmark Equilibrium) The model (20) - (22) under LIRE with the*

[6] In reference to Figure **??** this means that the negative root touches zero where it intersects with the hyperbola of the subspace condition.

endogenous information set $Z_t = \pi_t + \nu_t$ has an equilibrium with the properties

$$\begin{aligned} \pi_t &= \frac{1}{\phi - \rho} r_t - \nu_t, \\ r_{t|t} &= 0, \\ r_t &= \rho r_{t-1} + \varepsilon_t, \end{aligned} \tag{30}$$

where

$$\kappa_r = \phi - \rho, \Sigma = 0, \gamma_\varepsilon = 1/\left(\phi - \rho\right), \gamma_\nu = -1, \gamma_b = 0.$$

To summarize, in the LIRE model with an endogenous information set it is possible to find an equilibrium that (almost) replicates the FIRE equilibrium, subject to the presence of the measurement error. The scenario described in this section is akin to the outcome described in Lubik and Schorfheide (2003), where an indeterminate equilibrium without sunspots is observationally equivalent to a corresponding determinate equilibrium.

2.4.2 Variance Bounds

In our framework, we require any equilibrium to obey the central bank projections conditional on its nested information set., which guarantees that expectation formation of the two types of agents in the model is mutually consistent. As it turns out, however, this projection condition can also provide bound on the variance of the model's endogenous variables. In this section, we show specifically that the benchmark equilibrium discussed above has the highest inflation variance of all equilibria. Recall that the RE solutions in the space of the central bank projections is $\pi_{t|t} = \frac{1}{\phi - \rho} r_{t|t}$. This implies the projection condition $cov\left(\widetilde{\pi}_t, \widetilde{Z}_t\right) = \frac{1}{\phi-\rho} cov\left(\widetilde{r}_t, \widetilde{Z}_t\right)$ for information set Z_t. We showed above that there is no benchmark equilibrium in the case of an endogenous information set as the Kalman gain stays bounded within the unit circle. We therefore focus on the endogenous information set $Z_t = \pi_t + \nu_t$. Substituting this expression into the projection condition and expanding terms yields:

$$var(\widetilde{\pi}_t) + cov\left(\widetilde{\pi}_t, \nu_t\right) = \frac{1}{\phi - \rho} cov\left(\widetilde{\pi}_t, \widetilde{r}_t\right), \tag{31}$$

where we have made use of the fact that $cov\left(\widetilde{r}_t, \nu_t\right) = 0$. We can now collect terms and write:

$$var(\widetilde{\pi}_t) = cov\left(\widetilde{\pi}_t, \frac{1}{\phi-\rho}\widetilde{r}_t - \nu_t\right). \tag{32}$$

Using the Cauchy-Schwarz inequality this allows us to derive the upper bound on the inflation projection error variance:

$$var(\widetilde{\pi}_t) \leq var\left(\frac{1}{\phi-\rho}\widetilde{r}_t - \nu_t\right) = \left(\frac{1}{\phi-\rho}\right)^2 var(\widetilde{r}_t) + \sigma_\nu^2. \tag{33}$$

Since $\pi_t = \widetilde{\pi}_t + \pi_{t|t-1}$ and also $\pi_{t|t-1} = \frac{1}{\phi-\rho} r_{t|t-1}$, we can derive the expression:

$$var(\pi_t) = var(\widetilde{\pi}_t) + \left(\frac{1}{\phi-\rho}\right)^2 var(r_{t|t-1}) + 2cov\left(\widetilde{\pi}_t, r_{t|t-1}\right), \tag{34}$$

whereby the covariance is zero under optimal projections. Similarly, $var(r_t) = var(\widetilde{r}_t) + var(r_{t|t-1})$. Substituting these expressions and collecting terms then results in:

$$var(\pi_t) \leq \left(\frac{1}{\phi-\rho}\right)^2 var(r_t) + \sigma_\nu^2 = \sigma_\nu^2 + \frac{\sigma_\varepsilon^2}{(1-\rho)(\phi-\rho)^2}. \tag{35}$$

We note that the second term in the variance bound is the inflation variance under FIRE, whereby $\pi_t = \frac{1}{\phi-\rho} r_t$ and $var(\pi_t) = \frac{\sigma_\varepsilon^2}{(1-\rho)(\phi-\rho)^2}$, whereas the first term is the measurement error variance σ_ν^2. We showed above that the benchmark equilibrium in which the endogenous forecast error is uniquely determined is $\pi_t^* = \frac{1}{\phi-\rho} r_t - \nu_t$, with its variance given by $var(\pi_t^*) = \sigma_\nu^2 + \frac{\sigma_\varepsilon^2}{(1-\rho)(\phi-\rho)^2}$, which is equal to the upper bound in the expression above. We can therefore conclude that the inflation variance in the benchmark equilibrium is the highest inflation variance of any equilibria under LIRE with an endogenous information set, i.e., $var(\pi_t) \leq var(\pi_t^*)$. This bound applies to other equilibria if they exist. It does not presuppose that other such equilibria exist for every parameter configuration, but since all potential equilibria must satisfy the projection condition the bound applies.

2.4.3 MSV Solution: Svensson and Woodford (2004)

In this section we describe a minimum-state-variable (MSV) approach as in Svensson and Woodford (2004). An important difference between their work and ours is that, in the present model is described by a given rule, whereas Svensson and Woodford endevaour to characterize optimal policy. However, for a given set of first-order conditions to the optimal policy problem under imperfect information, their approach falls into the class of expectational linear-difference equations studied here as well (see Section **??** for a more general discussion). Svensson and Woodford (2004) are not alone in pursing a MSV approach in such models, other examples are given by Aoki (2008), or Nimark (2008); applied to our model, this approach begins with a guess that the equilibrium process for inflation has the following form:

$$\pi_t = g\, r_t^* + \bar{g}\, r_{t|t} \qquad \bar{g} \equiv \frac{1}{\phi - \rho} \tag{36}$$

$$= g\, r_t + (\bar{g} - g)\, r_{t|t} \tag{37}$$

For any choice of g, this guess automatically satisfies the sub-space condition $\pi_{t|t} = \bar{g} r_{t|t}$. What remains to be seen is which values for g (if any) would be consistent with the rest of the dynamic system, notably the innovations version of Fisher equation in (**??**). Notice that the proposed solution excludes belief shocks.

Let us proceed by deriving the dynamics for r_t^* and $r_{t|t}$ implied by (36) for a given value of g. A slight complication for setting up the Kalman filter — encountered also by Svensson and Woodford — is that the guess for inflation in (36) depends on the projected real rate, and thus on the history of measurements (Z^t) which in turn depends on the history of inflation:

$$Z_t = g\, r_t + (\bar{g} - g)\, r_{t|t} + \nu_t \tag{38}$$

However, notice that the term in $r_{t|t}$ does not add any new information to Z_t; in fact

Z_t rather provides an implicit definition of an information set spanned by:

$$W_t = g\, r_t + \nu_t \tag{39}$$

in the sense that $E(x_t|Z^t) = E(x_t|W^t)$ for any variable x_t. While projections of variables onto W^t and Z^t are equivalent, the associated Kalman gains will, however, differ by a factor of proportionality.[7]

For starters, consider the Kalman gain involved in projecting the real rate onto W^t, $\tilde{r}_{t|t} = \kappa \tilde{W}_t$:

and define

$$R^2 \equiv g \cdot K \qquad\qquad \Rightarrow \quad 0 \leq R^2 \leq 1. \tag{40}$$

We can then write

$$\tilde{\pi}_t = g \cdot \tilde{r}_t + (\bar{g} - g)\, \kappa\, \tilde{W}_t \tag{41}$$

$$= \left(g \cdot (1 - R^2) + \bar{g} \cdot R^2\right) \tilde{r}_t + (\bar{g} - g)\, \kappa\, \nu_t \tag{42}$$

$$= \left(g \cdot (1 - R^2) + \bar{g} \cdot R^2\right) \rho\, r^*_{t-1} + \underbrace{\left(g \cdot (1 - R^2) + \bar{g} \cdot R^2\right) \varepsilon_t + (\bar{g} - g)\, \kappa\, \nu_t}_{=\eta_t} \tag{43}$$

where the last line uses $\tilde{r}_t = \rho\, r^*_{t-1} + \varepsilon_t$.

In order to match (**??**) we can set η_t equal to the shock components of (43) as indicated above *and* we need to find a value for g that sets the loading on r^*_{t-1} in (43) equal to minus

[7] Let K_r continue to denote the Kalman gain of r_t onto Z_t and we have

$$\tilde{Z}_t = \tilde{W}_t + (\bar{g} - g) \cdot K_r \cdot \tilde{Z}_t = \tilde{W}_t / (1 - (\bar{g} - g) \cdot K_r).$$

one:

$$\left(g\cdot(1-R^2)+\bar{g}\cdot R^2\right)\rho=-1 \tag{44}$$

$$\Rightarrow \quad g\leq 0 \tag{45}$$

where the inequality follows from $\bar{g}$, R^2 and ρ being all positive numbers. As a further condition, the solution approach espoused by Svensson and Woodford (2004) would require the roots of the characteristic equation describing the joint dynamics of π_t, $r_{t|t}$ and r_t, see (20)- (22) above, to satisfy the usual counting rule for values inside and outside the unit circle. In the present case, with only one backward-looking variables, r_t, and two forward-looking variables, π_t and $r_{t|t}$, the approach of Svensson and Woodford (2004) would rely on finding one stable and two unstable eigenvalues. However, it can be shown that in the present example, the Kalman filter will always stabilize the dynamics of $r_t - r_{t|t}$ causing the system to have two stable and only one unstable root.

Note that the set of MSV candidate solutions — described by (36) for any given value of g — does not span the set of all candidate solutions that we have looked at so far — described by any combination of weights γ for the linear combination of shocks that make up the endogenous forecast error η_t: Furthermore, the set of SW candidates does not even span the restricted set of candidates for η_t where $\gamma_b = 0$. To see this, notice that the MSV candidate is parametrized by a single unknown coefficient, g, which places a restriction on the weights γ_ε and γ_ν implied by the associated specification of η_t as seen in (43).

2.5 Discussion

[TO BE WRITTEN]

3 General Setup

3.1 A linear RE system with asymmetric information

Denote the policy instrument by $\boldsymbol{i}_t$ and let $\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$ be vectors of backward- and forward-looking variables that do not include $\boldsymbol{i}_t$.[8]

The backward-looking variables are characterized by *exogenous* forecast errors, $\boldsymbol{\varepsilon}_t$:

$$\boldsymbol{X}_t - E_{t-1}\boldsymbol{X}_t = \boldsymbol{B}_{x\varepsilon}\, \boldsymbol{\varepsilon}_t \qquad\qquad \boldsymbol{\varepsilon}_t \sim N(\boldsymbol{0}, \boldsymbol{I}) \tag{46}$$

where the number of independent, exogenous shocks, N_ε may be smaller than the number of backward-looking variables, N_x, and $\boldsymbol{B}_{x\varepsilon}$ is assumed to have full column-rank. Forecast errors for the forward-looking variables, denoted

$$\boldsymbol{\eta}_t \equiv \boldsymbol{Y}_t - E_{t-1}\boldsymbol{Y}_t, \tag{47}$$

are endogenous and remain to be determined as part of the model's RE solution.

Folding $\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$ into the vector $\boldsymbol{S}_t$, we consider the following set of dynamic equations:

$$\boldsymbol{S}_t = \begin{bmatrix} \boldsymbol{X}_t \\ \boldsymbol{Y}_t \end{bmatrix} \tag{48}$$

$$\boldsymbol{J} E_t \boldsymbol{S}_{t+1} + \hat{\boldsymbol{J}} \boldsymbol{S}_{t+1|t} = \boldsymbol{H}\boldsymbol{S}_t + \hat{\boldsymbol{H}}\boldsymbol{S}_{t|t} + \boldsymbol{H}_i \boldsymbol{i}_t \qquad\qquad |\boldsymbol{J}| \neq 0 \tag{49}$$

$$\boldsymbol{i}_t = \boldsymbol{\Phi}_i \boldsymbol{i}_{t-1} + \boldsymbol{\Phi}_J \boldsymbol{S}_{t+1|t} + \boldsymbol{\Phi}_H \boldsymbol{S}_{t|t} \tag{50}$$

We require $|\boldsymbol{J}| \neq \boldsymbol{0}$, though it might be possible to extend our approach to handle also cases where $\boldsymbol{J}$ is singular. Typically, such cases arise in the presence of static, definitional equations, which could be substituted out to fit (49).

[8] Throughout, vectors and matrices will be denoted with bold letters; notice, however, that our use of lower- and uppercase letters does *not* distinguish between matrices and vectors. In most applications, $\boldsymbol{i}_t$ is likely to be a scalar, but nothing in our framework hinges on this assumption and so we use the generic vector notation, $\boldsymbol{i}_t$, throughout. In our context, keeping the policy instrument separate from $\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$ will be useful since $\boldsymbol{i}_t$ will always be assumed to be perfectly known and observable to both public and central bank.

The imperfectly informed policymaker — in light of our examples synonymously referred to as "central bank" — sets $\boldsymbol{i}_t$ according to the rule given in (50). By definition, the policymaker must know the current value and history of her instrument choices. Moreover, all variables entering (50) are expressed as expectations conditional on the central bank's information set, denoted $\boldsymbol{S}_{t+1|t}$ and $\boldsymbol{S}_{t|t}$. Before describing the nature of these expectations and the underlying informational assumptions in more detail, we would like to point out that the form of the reaction function for the policy instrument given in (50) can capture a variety of settings for policymaking. For example, as illustrated in our paper, such a reaction function can capture Taylor-type interest rate rules when evaluated using central-bank projections as studied, among others, by Orphanides (2001, 2003). As demonstrated in our Fisher-economy example, such a reaction function can also depend on exogenous driving variables via its dependence on $\boldsymbol{X}_t$ as part of $\boldsymbol{S}_t$.[9] Using suitable definitions of $\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$, the reaction function in (50) has also the form of optimal policy as prescribed by the first-order conditions for optimal policy under asymmetric information as derived by Svensson and Woodford (2004) and Aoki (2006) in a comparable setting.[10]

The policymaker is supposed to form rational expectations based on an information set that is characterized by the observed history of a signal, denoted $\boldsymbol{Z}_t$ (as well as knowledge of all model parameters). For any variable $\boldsymbol{V}_t$, and any lead or lag h, $E_t\boldsymbol{V}_{t+h}$ denotes expectations based on full information whereas

$$\boldsymbol{V}_{t+h|t} \equiv E(\boldsymbol{V}_{t+h}|\boldsymbol{Z}^t) \qquad\qquad \boldsymbol{Z}^t = \{\boldsymbol{Z}_t, \boldsymbol{Z}_{t-1}, \boldsymbol{Z}_{t-2}, \ldots\} \tag{51}$$

[9] In principle, the reaction function could also a feature an exogenous residual in the form of a "policy error" captured as part of $\boldsymbol{X}_t$ that enters the system only via (50). However, in most meaningful circumstances the policy error should be spanned by the central bank's information set. In light of our assumption that the central bank's information set is nested by the public's information set, realizations of this policy residual would thus end up being common knowledge. Except for the effects of asymmetric information, we study, however, a linear system and the effects of exogenous shocks that are common knowledge, will be identical to the full information case.

[10] Specifically, Svensson and Woodford (2004)augment a system similar to (49) and identical informational assumption as used here with a quadratic loss function to derive linear first-order conditions akin to (50); see their equations (15) and (40) for the cases of optimal policy under discretion and commitment, respectively. (In the commitment case, the vector of backward-looking variables would also have to include the evolution of Lagrange multipliers associated with equations describing the private sector's forward-looking behavior.)

denotes conditional expectations under the central bank information set. By construction, central bank actions, $\boldsymbol{i}_t$, are spanned by the history of observed signals, such that we always have $\boldsymbol{i}_t = \boldsymbol{i}_{t|t}$ even though the policy instrument will not be explicitly listed as part of the measurement vector $\boldsymbol{Z}_t$.

In our linear Gaussian setting, conditional expectations will be represented by linear projections that can be computed via the Kalman filter. Accordingly, we will refer to the central bank's expectations synonymously as "projections;" however, not without emphasizing that these are indented to represent optimal inference of the central bank under limited information. For further use, it will be helpful to introduce the following notation for innovations $\tilde{\boldsymbol{V}}_t$ and residuals $\boldsymbol{V}_t^*$:

$$\tilde{\boldsymbol{V}}_t \equiv \boldsymbol{V}_t - \boldsymbol{V}_{t|t-1} , \qquad \boldsymbol{V}_t^* \equiv \boldsymbol{V}_t - \boldsymbol{V}_{t|t} = \tilde{\boldsymbol{V}}_t - \tilde{\boldsymbol{V}}_{t|t} . \tag{52}$$

The measurement vector is generally given by a linear combination of backward- and forward-looking variables:[11]

$$\boldsymbol{Z}_t = \overline{\boldsymbol{C}}\boldsymbol{S}_t = \overline{\boldsymbol{C}}_x\boldsymbol{X}_t + \overline{\boldsymbol{C}}_y\boldsymbol{Y}_t \tag{53}$$

For concreteness, we delineate the following two cases: one where the signal depends on endogenous variables (specifically, choosing $\overline{\boldsymbol{C}}_y = \boldsymbol{I}$) as well as the case where the signal solely reflects exogenous variables ($\overline{\boldsymbol{C}}_y = \boldsymbol{0}$ and $\boldsymbol{X}_t$ exogenous). Both cases are described next; since both are nested by the general definition given by (53), we will continue to refer to (53), unless reference to a specific cases is necessary.

[11]Please recall that "measurement errors" — disturbances to the measurement equation that would otherwise be absent from a full-information version of the model — are assumed to have been lumped into the vector of backward-looking variables, $\boldsymbol{X}_t$. By construction, we have then $\boldsymbol{Z}_{t|t} = \overline{\boldsymbol{C}}\boldsymbol{S}_{t|t} = \boldsymbol{Z}_t$ and thus $\overline{\boldsymbol{C}}\,\text{Var}\left(\boldsymbol{S}_t|\boldsymbol{Z}^t\right)\overline{\boldsymbol{C}}' = \boldsymbol{0}$.

3.1.1 Endogenous signal

In (53), the signal observed by the central bank involves a linear combination of forward- and backward-looking variables, such that the signal depends at least in part on endogenous variables. When considering this case, and to simplify some of the algebra, we limit ourselves to signal vectors that have the same length as the vector of forward-looking variables ($\boldsymbol{Y}_t$) and that have no rank-deficient loading on $\boldsymbol{Y}_t$. All told, we assume that $\overline{\boldsymbol{C}}_y$ in (53) is square and invertible. In this case, $\overline{\boldsymbol{C}}_y$ can be normalized to the identity matrix.[12]

In the endogenous-signal case, we thus consider signal vectors of the form

$$\boldsymbol{Z}_t = \overline{\boldsymbol{C}}_x \boldsymbol{X}_t + \boldsymbol{Y}_t \qquad \text{and thus} \quad \overline{\boldsymbol{C}} = \begin{bmatrix} \overline{\boldsymbol{C}}_x & \boldsymbol{I} \end{bmatrix}. \tag{54}$$

Note that the endogenous-signal setup also includes the case where each forward-looking variable is observed with error, as in $\boldsymbol{Z}_t = \boldsymbol{Y}_t + \boldsymbol{\nu}_t$ where $\boldsymbol{\nu}_t$ is an exogenous measurement error to be included among the set of backward-looking variables in $\boldsymbol{X}_t$.

3.1.2 Exogenous signal

To consider the case of a purely exogenous signal, we need to distinguish between endogenous and exogenous components of the vector of backward-looking variables $\boldsymbol{X}_t$. Let $\boldsymbol{X}_t$ be partitioned into exogenous variables, denoted $\boldsymbol{x}_t$, and endogenous variables (like the lagged inflation rate in case of a Phillips Curve with indexation), denoted $\boldsymbol{k}_t$.[13]

Exogeneity of $\boldsymbol{x}_t$ places several zero restrictions on the system matrices in (49), and its dynamics are reduced to

$$\boldsymbol{x}_t = \boldsymbol{h}_{xx} \boldsymbol{x}_{t-1} + \boldsymbol{b}_{x\varepsilon} \boldsymbol{\varepsilon}_t \tag{55}$$

[12]Consider the case of a signal $\hat{\boldsymbol{Z}}_t = \hat{\boldsymbol{C}}_x \boldsymbol{X}_t + \hat{\boldsymbol{C}}_y \boldsymbol{Y}_t$ where $\hat{\boldsymbol{C}}_y$ is square and nonsingular. The information content provided by $\hat{\boldsymbol{Z}}_t$ is equivalent to what is spanned by $\boldsymbol{Z}_t = \hat{\boldsymbol{C}}_y^{-1} \hat{\boldsymbol{Z}}_t$ with $\overline{\boldsymbol{C}}_x = \hat{\boldsymbol{C}}_y^{-1} \hat{\boldsymbol{C}}_x$.

[13]The dynamics of $\boldsymbol{k}_t$ are not of concern for now. In general, we can think of their transition equation as $\boldsymbol{k}_{t+1} = \boldsymbol{h}_{kx} \boldsymbol{x}_t + \boldsymbol{h}_{kk} \boldsymbol{k}_t + \boldsymbol{h}_{ky} \boldsymbol{Y}_t + \boldsymbol{h}_{ki} i_t + \boldsymbol{B}_{k\varepsilon} \boldsymbol{\varepsilon}_{t+1}$.

where $\boldsymbol{h}_{xx}$ and $\boldsymbol{b}_{x\varepsilon}$ are appropriate sub-blocks of $\boldsymbol{H}$ and $\boldsymbol{B}_{x\varepsilon}$. The signal is then given by

$$\boldsymbol{Z}_t = \overline{\boldsymbol{C}}_x \boldsymbol{x}_t. \tag{56}$$

Our presentation will mainly focus on the endogenous-signal case and return to the case of an exogenous signal later in Section 3.3.4.

3.1.3 Mapping a few examples into the general setup

The simple Fisher-equation example, described in Section 2, can be cast into the general framework described here as follows:

$$\boldsymbol{X}_t = \begin{bmatrix} r_t \\ \nu_t \end{bmatrix} \quad \text{and} \quad \boldsymbol{Y}_t = \pi_t \tag{57}$$

The New Keynesian (NK) example described in Section 4 of the paper can fits into our general framework as follows:

$$\boldsymbol{X}_t = \begin{bmatrix} r_t \\ \Delta\bar{y}_t \\ \nu_t^{\pi} \\ \nu_t^{y} \\ \bar{y}_{t-1} \\ \pi_{t-1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{Y}_t = \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} \tag{58}$$

3.1.4 Full information solution

The full-information system can easily be solved using familiar methods, known, for example from Klein (2000) or King and Watson (1998). We stack all variables, including the policy control, into a vector $\boldsymbol{\mathcal{S}}_t$ that is is partitioned into a vector of $N_i + N_x$ backward-looking variables, $\boldsymbol{\mathcal{X}}_t$, and a vector of $N_y + N_i$ forward-looking variables, $\boldsymbol{\mathcal{Y}}_t$:[14]

$$\boldsymbol{\mathcal{S}}_t = \begin{bmatrix} \boldsymbol{\mathcal{X}}_t \\ \boldsymbol{\mathcal{Y}}_t \end{bmatrix} \qquad \text{where} \quad \boldsymbol{\mathcal{X}}_t = \begin{bmatrix} \boldsymbol{i}_{t-1} \\ \boldsymbol{X}_t \end{bmatrix} \qquad \boldsymbol{\mathcal{Y}}_t = \begin{bmatrix} \boldsymbol{Y}_t\ , \\ \boldsymbol{i}_t \end{bmatrix} \tag{59}$$

Using, $\boldsymbol{\mathcal{S}}_t' = \begin{bmatrix} \boldsymbol{i}_{t-1}' & \boldsymbol{S}_t' & \boldsymbol{i}_t' \end{bmatrix}$, the dynamics of the system under full information are then characterized by the following expectational difference equation:

$$\underbrace{\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J} + \hat{\boldsymbol{J}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Phi}_J & \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{\mathcal{J}}} E_t \boldsymbol{\mathcal{S}}_{t+1} = \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{H} + \hat{\boldsymbol{H}} & \boldsymbol{H}_i \\ -\boldsymbol{\Phi}_i & -\boldsymbol{\Phi}_H & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{\mathcal{A}}} E_t \boldsymbol{\mathcal{S}}_t \tag{60}$$

Throughout, we assume that the pencil $|\boldsymbol{\mathcal{J}} - \boldsymbol{\mathcal{A}} z|$ has $N_i + N_x$ roots outside the unit circle and $N_y + N_i$ roots inside the unit circle.[15] This condition ensures a unique equilibrium under full information, and the solution has the following form:

$$E_t \boldsymbol{\mathcal{X}}_{t+1} = \boldsymbol{\mathcal{P}}\ \boldsymbol{\mathcal{X}}_t \qquad\qquad \boldsymbol{\mathcal{Y}}_t = \boldsymbol{\mathcal{G}}\ \boldsymbol{\mathcal{X}}_t \tag{61}$$

which can be broken down further into

$$E_t \boldsymbol{X}_{t+1} = \boldsymbol{\mathcal{P}}_{xx}\ \boldsymbol{X}_t + \boldsymbol{\mathcal{P}}_{xi}\ \boldsymbol{i}_{t-1} \tag{62}$$

$$\boldsymbol{i}_t = \boldsymbol{\mathcal{P}}_{ix}\ \boldsymbol{X}_t + \boldsymbol{\mathcal{P}}_{ii}\ \boldsymbol{i}_{t-1} \tag{63}$$

$$\boldsymbol{Y}_t = \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{X}_t + \boldsymbol{\mathcal{G}}_{yi}\ \boldsymbol{i}_{t-1} \tag{64}$$

[14] The presence of the lagged policy control in $\boldsymbol{\mathcal{X}}_t$ serves to handle the case of interest-rate smoothing, $\boldsymbol{\Phi}_i \neq 0$, and can otherwise be omitted. In the case of interest rate smoothing, $\boldsymbol{i}_{t-1}$ enters the system as a backward-looking variables. In the setups of Klein (2000) or King and Watson (1998), it is required that all backward-looking variables be placed at the top of $\boldsymbol{\mathcal{S}}_t$.

[15] In the case of the simple Fisher economy in Section 2, the root-counting condition was satisfied by requiring that the central bank's interest-rate rule satisfied the Taylor principle, responding more than one-to-one to fluctuations in inflation.

$$\text{with} \quad \boldsymbol{\mathcal{P}} = \begin{bmatrix} \boldsymbol{\mathcal{P}}_{ii} & \boldsymbol{\mathcal{P}}_{ix} \\ \boldsymbol{\mathcal{P}}_{ix} & \boldsymbol{\mathcal{P}}_{xx} \end{bmatrix} \qquad \text{and} \quad \boldsymbol{\mathcal{G}} = \begin{bmatrix} \boldsymbol{\mathcal{G}}_{yi} & \boldsymbol{\mathcal{G}}_{yx} \\ \boldsymbol{\mathcal{G}}_{ii} & \boldsymbol{\mathcal{G}}_{ix} \end{bmatrix} . \tag{65}$$

For future reference it will also be convenient to define $\boldsymbol{\mathcal{P}}_i$, $\boldsymbol{\mathcal{G}}_y$, $\boldsymbol{\mathcal{I}}$, and $\boldsymbol{\mathcal{F}}$ such that:[16]

$$\boldsymbol{i}_t = \boldsymbol{\mathcal{P}}_i \, \boldsymbol{\mathcal{X}}_t \qquad \boldsymbol{\mathcal{P}}_i = \begin{bmatrix} \boldsymbol{\mathcal{P}}_{ii} & \boldsymbol{\mathcal{P}}_{ix} \end{bmatrix} \tag{66}$$

$$\boldsymbol{Y}_t = \boldsymbol{\mathcal{G}}_y \, \boldsymbol{\mathcal{X}}_t \qquad \boldsymbol{\mathcal{G}}_y = \begin{bmatrix} \boldsymbol{\mathcal{G}}_{yi} & \boldsymbol{\mathcal{G}}_{yx} \end{bmatrix} \tag{67}$$

$$\boldsymbol{X}_t = \boldsymbol{\mathcal{I}} \, \boldsymbol{\mathcal{X}}_t \qquad \boldsymbol{\mathcal{I}} \equiv \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{68}$$

$$\boldsymbol{S}_t = \boldsymbol{\mathcal{F}} \boldsymbol{\mathcal{X}}_t \qquad \boldsymbol{\mathcal{F}} = \begin{bmatrix} \boldsymbol{\mathcal{I}} \\ \boldsymbol{\mathcal{G}}_y \end{bmatrix} \tag{69}$$

Equilibrium dynamics in the full-information case are then be summarized by:

$$\boldsymbol{\mathcal{X}}_{t+1} = \boldsymbol{\mathcal{P}} \boldsymbol{\mathcal{X}}_t + \boldsymbol{B}_{x\varepsilon} \boldsymbol{\varepsilon}_{t+1} \qquad \boldsymbol{\mathcal{B}} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{B}_{x\varepsilon} \end{bmatrix} \tag{70}$$

$$\boldsymbol{\mathcal{Y}}_t = \boldsymbol{\mathcal{G}} \boldsymbol{\mathcal{X}}_t \tag{71}$$

The endogenous forecast errors are given by $\eta_t = \boldsymbol{\mathcal{G}}_{yx} \, \boldsymbol{B}_{x\varepsilon} \, \varepsilon_t$. The decision-rule coefficients $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{G}}$ do not depend on the shock variances encoded in $\boldsymbol{B}_{x\varepsilon}$ and, of course, not on the measurement loadings $\overline{\boldsymbol{C}}$ either.

Measurement errors that are included in $\boldsymbol{X}_t$ are, of course, superfluous for a full-information solution. The measurement errors would affect endogenous variables of the system only via $\overline{\boldsymbol{C}}$, which does not play a role in the full information solution. But, also when computing a full-information solution, there is no harm including measurement errors in $\boldsymbol{X}_t$: The corresponding columns of $\boldsymbol{\mathcal{G}}_{yx}$ — as generated, for example, by the procedures of Klein (2000) or King and Watson (1998)— are set to zero in this case.

[16] Note further that the definitions of $\boldsymbol{\mathcal{X}}_t$ and $\boldsymbol{\mathcal{Y}}_t$ imply $\boldsymbol{\mathcal{G}}_{ii} = \boldsymbol{\mathcal{P}}_{ii}$ and $\boldsymbol{\mathcal{G}}_{ix} = \boldsymbol{\mathcal{P}}_{ix}$.

3.2 System conditioned down on Central Bank Projections

Conditioning all variables down onto $\boldsymbol{Z}^t$ we get the same linear RE system as under full information, just in terms of projections onto $\boldsymbol{Z}^t$:

$$\boldsymbol{\mathcal{J}}\,\boldsymbol{\mathcal{S}}_{t+1|t} = \boldsymbol{\mathcal{A}}\,\boldsymbol{\mathcal{S}}_{t|t} \tag{72}$$

and the solution mimics the form known from the full-information case

$$\boldsymbol{\mathcal{X}}_{t+1|t} = \boldsymbol{\mathcal{P}}\;\boldsymbol{\mathcal{X}}_{t|t} \qquad\qquad \boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}}\;\boldsymbol{\mathcal{X}}_{t|t} \tag{73}$$

which — recalling that $\boldsymbol{i}_t = \boldsymbol{i}_{t|t} = \boldsymbol{i}_{t|t+1}$ — can again be further broken down into

$$\boldsymbol{X}_{t+1|t} = \boldsymbol{\mathcal{P}}_{xx}\;\boldsymbol{X}_{t|t} + \boldsymbol{\mathcal{P}}_{xi}\;\boldsymbol{i}_{t-1} \qquad\qquad \boldsymbol{Y}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\;\boldsymbol{X}_{t|t} + \boldsymbol{\mathcal{G}}_{xi}\;\boldsymbol{i}_{t-1} \quad \text{etc.} \tag{74}$$

Anticipating a linear Gaussian equilibrium, optimal expectations of the central bank can be represented by a Kalman filter (details to be described further below). Among others, the Kalman filter implies a linear relationship between projected innovations of unobserved variables and observed innovations in the signal, as in $\tilde{\boldsymbol{\mathcal{X}}}_{t|t} = \boldsymbol{\mathcal{K}}_{\mathcal{X}}\tilde{\boldsymbol{Z}}_{t+1}$ where $\boldsymbol{\mathcal{K}}_{\mathcal{X}}$ is a yet to be derived Kalman gain. Conditional on a sequence of innovations to the central bank's information set — that is a sequence of $\tilde{\boldsymbol{Z}}_t$ — central bank projections evolve according to:

$$\boldsymbol{\mathcal{X}}_{t+1|t+1} = \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{X}}_{t|t} + \tilde{\boldsymbol{\mathcal{X}}}_{t+1|t+1} \qquad \tilde{\boldsymbol{\mathcal{X}}}_{t+1|t+1} = \boldsymbol{\mathcal{K}}_{\mathcal{X}}\tilde{\boldsymbol{Z}}_{t+1} \qquad \boldsymbol{\mathcal{K}}_{\mathcal{X}} = \begin{bmatrix} 0 \\ K_x \end{bmatrix} \tag{75}$$

(The upper block of $\boldsymbol{\mathcal{K}}_{\mathcal{X}}$ is zero since $\boldsymbol{i}_{t-1|t} = \boldsymbol{i}_{t-1|t-1} = \boldsymbol{i}_{t-1}$.)

Since we assumed that conditions for a unique rational expectations equilibrium under full information are satisfied, $\boldsymbol{\mathcal{P}}$ is a stable matrix. Consequently, for any bounded sequence of $\tilde{\boldsymbol{Z}}_t$, the dynamics of central bank projections for $\boldsymbol{\mathcal{X}}_{t|t}$ as well as $\boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{X}}_{t|t}$ are thus stable, producing only bounded outcomes for $\boldsymbol{\mathcal{X}}_{t|t}$ and $\boldsymbol{\mathcal{Y}}_{t|t}$.[17]

3.2.1 The projection condition

Equation (74) is an equilibrium condition that restricts the dynamics of central bank projections; henceforth we refer to the resulting restrictions as "projection condition." The projection condition has implications for innovation dynamics and the Kalman gains. Noting that $\boldsymbol{i}_{t-1} = \boldsymbol{i}_{t-1|t-1}$ we obtain the following restrictions between projections of innovations in forward- and backward-looking variables:

$$\tilde{\boldsymbol{Y}}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\ \tilde{\boldsymbol{X}}_{t|t} \qquad \text{and thus} \quad \operatorname{Cov}(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t) = \boldsymbol{\mathcal{G}}_{yx}\ \operatorname{Cov}(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t) \tag{76}$$

Alternatively, denoting Kalman gains by

$$\boldsymbol{K}_x = \operatorname{Cov}(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t)\left(\operatorname{Var}(\tilde{\boldsymbol{Z}}_t)^{-1}\right) \tag{77}$$

$$\boldsymbol{K}_y = \operatorname{Cov}(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t)\left(\operatorname{Var}(\tilde{\boldsymbol{Z}}_t)^{-1}\right) \tag{78}$$

$$\boldsymbol{K}_i = \operatorname{Cov}(\tilde{\boldsymbol{i}}_t, \tilde{\boldsymbol{Z}}_t)\left(\operatorname{Var}(\tilde{\boldsymbol{Z}}_t)^{-1}\right) \tag{79}$$

the projection condition implies

$$\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\ \boldsymbol{K}_x \qquad\qquad \boldsymbol{K}_i = \boldsymbol{\mathcal{P}}_{ix}\ \boldsymbol{K}_x\ .. \tag{80}$$

The projection condition imposes a *second-moment* restriction on the joint distribution of the innovations $\tilde{\boldsymbol{X}}_t$ and $\tilde{\boldsymbol{Y}}_t$ that will be derived as part of the Kalman filter derived further below. Given the definition of the signal, these innovations also determine $\tilde{\boldsymbol{Z}}_t = \overline{\boldsymbol{C}}_x\tilde{\boldsymbol{X}}_t + \tilde{\boldsymbol{Y}}_t$ as well as $\tilde{\boldsymbol{i}}_{\boldsymbol{t}} = \tilde{\boldsymbol{i}}_{t|t} = \boldsymbol{\mathcal{P}}_{ix}\ \boldsymbol{K}_x\tilde{\boldsymbol{Z}}_t$.[18] As a second-moment restriction, the projection condition restricts only co-movements *on average* but not for any particular realization of $\tilde{\boldsymbol{X}}_t$ and $\tilde{\boldsymbol{Y}}_t$.

[17] As will be shown below, existence of a time-invariant, steady state Kalman filter, also implies stationarity of $\tilde{\boldsymbol{Z}}_t$. For now, recall that, by definition, $\tilde{\boldsymbol{Z}}_{t+1}$ is unpredictable under $\boldsymbol{Z}^t$: $\tilde{\boldsymbol{Z}}_{t|t-1} = 0$.

[18] Recall that $\tilde{\boldsymbol{i}}_t = \tilde{\boldsymbol{i}}_{t|t}$ holds automatically, since the policy instrument is defined to lie in the central bank's information set.

Substituting the projection condition (74) back into (49) yields the following:

$$E_t \boldsymbol{S}_{t+1} = \boldsymbol{A}\boldsymbol{S}_t + \hat{\boldsymbol{A}}_{S\mathcal{X}} \boldsymbol{\mathcal{X}}_{t|t} \tag{81}$$

$$\text{where} \quad \boldsymbol{A} \equiv \boldsymbol{J}^{-1}\boldsymbol{H} \ , \tag{82}$$

$$\hat{\boldsymbol{A}}_{S\mathcal{X}} = \boldsymbol{J}^{-1}\left(\hat{\boldsymbol{H}}\boldsymbol{\mathcal{F}} - \hat{\boldsymbol{J}}\boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{P}} + \boldsymbol{H}_i\boldsymbol{\mathcal{P}}_i\right) \ . \tag{83}$$

Compared to (49), the expectational difference system in (81) features no more forward-looking projections ($\boldsymbol{S}_{t+1|t}$) and the effects of policy are subsumed in projections of backward-looking variables ($\boldsymbol{\mathcal{X}}_{t|t}$). However, as before, this difference system still involves two different sets of expectations; these will be separated in the next step described below.

3.3 Innovations System

What remains to be determined are the dynamics of innovations, i.e. projections of current variables off the lagged history of central bank signals, i.e. $\tilde{\boldsymbol{S}}_t \equiv \boldsymbol{S}_t - \boldsymbol{S}_{t|t-1}$.[19] To characterize the innovation dynamics, we return to the original linear difference system given by (81), subtract the central bank projections, $\boldsymbol{S}_{t+1|t} = \boldsymbol{A}\boldsymbol{S}_{t|t} + \hat{\boldsymbol{A}}_{S\mathcal{X}}\boldsymbol{\mathcal{X}}_{t|t}$, from both sides of the system, and obtain the following *quasi*-difference system:

$$\tilde{\boldsymbol{S}}_{t+1} = \boldsymbol{A}(\tilde{\boldsymbol{S}}_t - \tilde{\boldsymbol{S}}_{t|t}) + \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_{t+1} \\ \boldsymbol{\eta}_{t+1} \end{bmatrix} \tag{84}$$

We refer to (84) as "quasi-difference" system since it describes the transition from projection *residuals* $\boldsymbol{S}^*_t = \tilde{\boldsymbol{S}}_t - \tilde{\boldsymbol{S}}_{t|t}$ to *innovations* $\tilde{\boldsymbol{S}}_t$.

Assuming an equilibrium with time-invariant, linear relationships and Gaussian disturbances, endogenous forecast errors can always be expressed as a linear combination of the exogenous shocks, $\boldsymbol{\varepsilon}_t$ — henceforth also referred to as "fundamental shocks" — and a set of N_y shocks that are orthogonal to these fundamental shocks, which we will refer to as "belief shocks, $\boldsymbol{b}_t$"; see also Farmer et al. (2015) and Lubik and Schorfheide (2003). While those

[19] Please recall the distinction between $\boldsymbol{S}_t$ and $\boldsymbol{\mathcal{S}}_t$: $\boldsymbol{\mathcal{S}}_t$ includes $\boldsymbol{S}_t$ and the policy control, which always lies in the span of $\boldsymbol{Z}^t$.

studies could merely rely on belief shocks to be uncorrelated with fundamentals and unanticipated, $E_{t-1}\boldsymbol{b}_t = 0$, we also need to assume that they are jointly normally distributed with the fundamental shocks:

$$\boldsymbol{\eta}_t = \boldsymbol{B}_{\eta\varepsilon}\,\boldsymbol{\varepsilon}_t + \boldsymbol{B}_{\eta b}\,\boldsymbol{b}_t \qquad\qquad \text{where} \quad \boldsymbol{b}_t \sim N(\mathbf{0}, \boldsymbol{I}) \tag{85}$$

As only the product $\boldsymbol{B}_{\eta b}\,\boldsymbol{b}_t$ enters the system, we normalize $\boldsymbol{b}_t$ to have a variance-covariance matrix equal to the identity matrix.

The coefficient matrices $\boldsymbol{B}_{\eta\varepsilon}$, and $\boldsymbol{B}_{\eta b}$ remain to be determined. Before turning to procedures that solves for valid values of those coefficients in Sections 3.3.3 and 3.3.4, we describe the conditions for valid values for these undetermined coefficients resulting in a time-invariant, equilibrium with stable dynamics.

A valid equilibrium with time-invariant linear decision rules requires that the shock loadings $\boldsymbol{B}_{\eta\varepsilon}$, and $\boldsymbol{B}_{\eta b}$ ensure the existence of a steady-state Kalman filter that also satisfies the projection condition as captured in the following definition.[20]

DEFINITION 1 (Equilibrium). *In a stable, linear, and time-invariant equilibrium, the forecast errors of the forward-looking variables are a linear combination of fundamental shocks and belief shocks, with time-invariant loadings* $\boldsymbol{B}_{\eta\varepsilon}$ *and* $\boldsymbol{B}_{\eta b}$*, as in* (85)*, and the belief shocks are normally distributed. Equilibrium dynamics of the forward-looking variables,* $\boldsymbol{Y}_t$*, satisfy the expectational difference system described by* (49) *and* (50)*. All expectations are rational, and the imperfectly informed agent's information set is described by* (53)*. In particular, in such an equilibrium, . . .*

1. *forward- and backward-looking variables are stationary,*

2. *a steady-state Kalman filter exists that characterizes central bank expectations conditional on the information set* $\boldsymbol{Z}^t$*; henceforth, simply referred to as "Kalman filter,"*

3. *the projection condition, stated in equation* (74)*, or equivalently in* (80)*, is satisfied.*

[20]Henceforth, references to valid equilibria are always understood in the context of time-invariant, linear decision rules and Gaussian belief shocks.

As will be shown next, the first two conditions listed in Definition 1 are satisfied as long as — for given values for the mappings from exogenous to endogenous shocks, encoded in $\boldsymbol{B}_{\eta\varepsilon}$, and $\boldsymbol{B}_{\eta b}$ — the residual system (84) and the measurement equation meet well-established conditions known from linear systems theory that are known as "detectability" and "stabilizability." Moreover, existence of a steady-state Kalman filter ensures also stable dynamics for innovations $\boldsymbol{\mathcal{S}}_t$. We then turn to an algorithm to solves for shock loadings that also satisfy the projection condition, as required by Definition 1 as well.

3.3.1 Kalman filter

In this section, we consider the necessary conditions to be imposed on the endogenous forecast errors $\boldsymbol{\eta}_t$, in order to assure the existence of a steady-state Kalman filter. So far, we do not enforce the projection condition, yet. Given a solution for $\boldsymbol{\eta}_t$ of the form in (85), exogenous and endogenous shocks have a joint, multivariate normal distribution. In this case, the Kalman filter describes optimal expectations. The presence of projected variables, $\boldsymbol{S}_{t|t}$, $\boldsymbol{S}_{t+1|t}$, $\boldsymbol{i}_t$ etc. in (49), does not affect the core of the central bank's filtering problem. Based on (84) and (53), and recalling $\boldsymbol{S}_t^* \equiv \tilde{\boldsymbol{S}}_t - \tilde{\boldsymbol{S}}_{t|t}$, we obtain the following state and measurement equations in innovations form:[21]

$$\tilde{\boldsymbol{S}}_{t+1} = \boldsymbol{A}\boldsymbol{S}_{t-1}^* + \boldsymbol{B}\boldsymbol{w}_{t+1} \qquad \boldsymbol{w}_{t+1} \equiv \begin{bmatrix} \boldsymbol{\varepsilon}_{t+1} \\ \boldsymbol{b}_{t+1} \end{bmatrix} \sim N(\boldsymbol{0}, \boldsymbol{I}) \tag{86}$$

$$\tilde{\boldsymbol{Z}}_{t+1} = \boldsymbol{C}\boldsymbol{S}_t^* + \boldsymbol{D}\boldsymbol{w}_{t+1} \tag{87}$$

$$\text{where} \quad \boldsymbol{B} \equiv \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & \boldsymbol{0} \\ \boldsymbol{B}_{\eta\varepsilon} & \boldsymbol{B}_{\eta b} \end{bmatrix}, \qquad \boldsymbol{C} = \overline{\boldsymbol{C}}\boldsymbol{A}, \qquad \boldsymbol{D} = \overline{\boldsymbol{C}}\boldsymbol{B}. \tag{88}$$

The state equation (86) and measurement equation (87) transform the innovations system provided by (84) and (53) into a system with explicit shocks to the measurement equation,

[21] Notice that the same dynamic system for innovations as in (86) and (87) would also result from a hypothetical state space given by $\boldsymbol{S}_{t+1} = \boldsymbol{A}\boldsymbol{S}_t + \boldsymbol{B}\boldsymbol{w}_{t+1}$ and $\boldsymbol{Z}_t = \overline{\boldsymbol{C}}\boldsymbol{S}_t$ with an identical Riccati equation for $\mathrm{Var}\,(\boldsymbol{S}_t|\boldsymbol{Z}^t)$ and identical Kalman gain; see also Hansen and Sargent (2005), Baxter et al. (2011) or Mertens (2016) for related arguments.

as captured by the term $\boldsymbol{D w_t}$ in (87); in this form we can state the following assumption that allows us to appeal to well-established results from Kalman-filtering theory about the existence of a stable solution.[22]

ASSUMPTION 1 (Non-degenerate shocks to the signal equation). *We assume that shocks to the signal equation have a full-rank variance-covariance matrix; that is* $|\boldsymbol{D}\boldsymbol{D}'| \neq 0$.

Existence of a steady-state Kalman filter relies on finding an ergodic distribution for $\boldsymbol{S}_t^*$ (and thus $\tilde{\boldsymbol{S}}_t$) with constant second moments $\boldsymbol{\Sigma} \equiv \mathrm{Var}\left(\boldsymbol{S}_t^*\right)$. When a steady-state filter exists, a constant Kalman gain, $\boldsymbol{K}$ relates projected innovations of $\tilde{\boldsymbol{S}}_t$ to innovations in the signal:

$$\tilde{\boldsymbol{S}}_{t|t} = \boldsymbol{K}\tilde{\boldsymbol{Z}}_t \tag{89}$$

$$\text{where} \quad \boldsymbol{K} = \mathrm{Cov}\left(\tilde{\boldsymbol{S}}_t, \tilde{\boldsymbol{Z}}_t\right)\left(\mathrm{Var}\left(\tilde{\boldsymbol{Z}}_t\right)\right)^{-1} = \left(\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}'\right)\left(\boldsymbol{C}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'\right)^{-1} \tag{90}$$

The dynamics of $\boldsymbol{S}_t^*$ are then characterized by

$$\boldsymbol{S}_{t+1}^* = \left(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}\right)\, \boldsymbol{S}_t^* + \left(\boldsymbol{B} - \boldsymbol{K}\boldsymbol{D}\right)\boldsymbol{w}_{t+1} \tag{91}$$

Existence of a steady-state filter depends on finding a symmetric, positive (semi) definite solution $\boldsymbol{\Sigma}$ to the following Riccati equation:

$$\begin{aligned}
\boldsymbol{\Sigma} &= (\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})\boldsymbol{\Sigma}(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C})' + \boldsymbol{B}\boldsymbol{B}' \\
&= \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}' + \boldsymbol{B}\boldsymbol{B}' - \boldsymbol{K}\left(\boldsymbol{C}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'\right)\boldsymbol{K}' \\
&= \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}' + \boldsymbol{B}\boldsymbol{B}' - \left(\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}'\right)\left(\boldsymbol{C}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{D}\boldsymbol{D}'\right)^{-1}\left(\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{C}' + \boldsymbol{B}\boldsymbol{D}'\right)'
\end{aligned} \tag{92}$$

Intuitively, the Kalman filter seeks to construct mean-squared error optimal projections $\boldsymbol{S}_{t|t}$ and seeks to minimize $\mathrm{Var}\left(\boldsymbol{S}_t^*\right)$. A necessary condition for the existence of a solution

[22] Alluding to jargon known from the work of Fernández-Villaverde et al. (2007), the system provided by (84) and (53) has "ABC" form while the system given by (86) and (87) has "ABCD" form. See also Komunjer and Ng (2011).

to this minimization problem is the ability to find at least some gain $\hat{\boldsymbol{K}}$ for which $\boldsymbol{A} - \hat{\boldsymbol{K}}\boldsymbol{C}$ is stable. Thus, when existence of the second moment for the residuals, $\text{Var}\left(\boldsymbol{S}_t^*\right) = \boldsymbol{\Sigma} \geq \boldsymbol{0}$, implies that the transition matrix in (91), $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$, is stable.

Formal conditions for the existence of a time-invariant Kalman filter have been stated, among others, by Anderson and Moore (1979), Anderson et al. (1996), Kailath et al. (2000), and Hansen and Sargent (2007). Necessary and sufficient conditions for the existence of a unique and stabilizing solution that is also positive semi-definite depend on the "detectability" and "unit-circle controllability" of certain matrices in our state space. We restate those concepts next.

DEFINITION 2 (Detectability). *A pair of matrices* $(\boldsymbol{A}, \boldsymbol{C})$ *is detectable when no right eigenvector of* $\boldsymbol{A}$ *that is associated with an unstable eigenvalue is orthogonal to the row space of* $\boldsymbol{C}$*. That is, there is no non-zero column vector* $\boldsymbol{v}$ *such that* $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{v}\lambda$ *and* $|\lambda| > 1$ *with* $\boldsymbol{C}\boldsymbol{v} = \boldsymbol{0}$.

Detectability alone is already sufficient for the existence of *some* solution to the Riccati equation such that $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ is stable (Kailath et al., 2000, Table E.1). Evidently, detectability is assured when $\boldsymbol{A}$ is a stable matrix, regardless of $\boldsymbol{C}$. To gain further intuition for the role of detectability, consider transforming $\boldsymbol{S}_t$ into "canonical variables" — as in the literature on solving rational expectations models (Blanchard and Kahn, 1980; King and Watson, 1998; Klein, 2000; Sims, 2002) — by premultiplying $\boldsymbol{S}_t$ with the matrix of eigenvectors of $\boldsymbol{A}$. Detectability then requires the signal equation (87) to have non-zero loadings on all unstable canonical variables for a stabilizing solution to (86) to exist.[23]

In order to consider the role of unit-circle controllability, it is useful to define the following two matrices:[24]

[23]Specifically, let $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$ with $\boldsymbol{\Lambda}$ diagonal be the eigenvalue-eigenvector factorization of $\boldsymbol{A}$ so that the columns of $\boldsymbol{V}$ correspond to the right eigenvectors of $\boldsymbol{A}$. Define canonical variables $\boldsymbol{S}_t^C \equiv \boldsymbol{V}^{-1}\boldsymbol{S}_t$. The signal equation can then be stated as $\boldsymbol{Z}_t = \boldsymbol{C}\boldsymbol{V}\boldsymbol{S}_t^C$ and detectability requires the signal equation to have non-zero loadings on at least every canonical variable associated with an unstable eigenvalue in $\boldsymbol{\Lambda}$.

[24]These transformation are designed to handle correlation between the shocks to signal and state equation in (86) and (87), that arises when $\boldsymbol{B}\boldsymbol{D}' \neq \boldsymbol{0}$.

$$\boldsymbol{A}^C \equiv \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}'\left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1}\boldsymbol{C} \qquad\qquad \boldsymbol{B}^C \equiv \boldsymbol{B}\underbrace{\left(\boldsymbol{I} - \boldsymbol{D}'\left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1}\boldsymbol{D}\right)}_{\mathcal{M}^D} \tag{93}$$

Notice that $\mathcal{M}^D$ is a projection matrix, which is symmetric and idempotent, $\mathcal{M}^D = \mathcal{M}^D\mathcal{M}^D$.[25] In our particular case, with $\boldsymbol{C} = \overline{\boldsymbol{C}}\boldsymbol{A}$ and $\boldsymbol{D} = \overline{\boldsymbol{C}}\boldsymbol{B}$, these expressions can be transformed further into

$$\boldsymbol{A}^C = (\boldsymbol{I} - \boldsymbol{P}^C)\boldsymbol{A} \quad \text{and} \quad \boldsymbol{B}^C = (\boldsymbol{I} - \boldsymbol{P}^C)\boldsymbol{B} \quad \text{where} \quad \boldsymbol{P}^C \equiv \boldsymbol{B}\overline{\boldsymbol{C}}'\left(\overline{\boldsymbol{C}}\boldsymbol{B}\boldsymbol{B}'\overline{\boldsymbol{C}}'\right)^{-1}\overline{\boldsymbol{C}}. \tag{94}$$

$\boldsymbol{P}^C$ is a non-symmetric, idempotent projection matrix with $\overline{\boldsymbol{C}}\boldsymbol{P}^C = \overline{\boldsymbol{C}}$.[26]

DEFINITION 3 (Unit-circle controllability). *The pair* $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ *is unit-circle controllable when no left-eigenvector of* $\boldsymbol{A}^C$ *associated with an eigenvalue on the unit circle is orthogonal to the column space of* $\boldsymbol{B}^C$*. That is, there is no non-zero row vector* $\boldsymbol{v}$ *such that* $\boldsymbol{v}\boldsymbol{A}^C = \boldsymbol{v}\lambda$ *with* $|\lambda| = 1$ *and* $\boldsymbol{v}\boldsymbol{B}^C = \boldsymbol{0}$*.*

THEOREM 1 (Stabilizing Solution to Riccati equation). *Provided Assumption 1 holds, a stabilizing and positive semi-definite solution to the Riccati equation* (92) *exists when* $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ *is unit-circle controllable and* $(\boldsymbol{A}, \boldsymbol{C})$ *is detectable. The steady-state Kalman gain is such that* $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ *is a stable matrix; moreover, the stabilizing solution is unique.*[27]

Proof. See Theorem E.5.1 of Kailath et al. (2000); related results are also presented in Anderson et al. (1996), or Chapter 4 of Anderson and Moore (1979). ∎

In our context, with $\boldsymbol{C} = \overline{\boldsymbol{C}}\boldsymbol{A}$ and $\boldsymbol{D} = \overline{\boldsymbol{C}}\boldsymbol{B}$, unit-circle controllability of $(\boldsymbol{A}^C, \boldsymbol{B}^C)$ is equivalent to unit-circle controllability of $(\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C), \boldsymbol{B})$. To see this, let $\tilde{\boldsymbol{v}} \equiv (\boldsymbol{I} - \boldsymbol{P}^C)\boldsymbol{v}$

[25] To appreciate the role of $\mathcal{M}^D$, consider the following thought experiment: $\mathcal{M}^D$ construct the residual in projecting the shocks of the system *off* the shocks in the signal equation, $\boldsymbol{w}_t - E(\boldsymbol{w}_t|\boldsymbol{D}\boldsymbol{w}_t) = \mathcal{M}^D\boldsymbol{w}_t$.

[26] An idempotent matrix is equal to its own square, that is $\boldsymbol{P}^C = \boldsymbol{P}^C\boldsymbol{P}^C$, and the eigenvalues of an idempotent matrix are either zero or one and we have $|\boldsymbol{P}^C| = \boldsymbol{0}$.

[27] There may be other, non-stabilizing positive semi-definite solutions.

and note that left-eigenvectors of $\boldsymbol{A}^C$ associated with eigenvalues on the unit circle cannot be orthogonal to $\boldsymbol{P}^C$ (otherwise we would have $\boldsymbol{v}\boldsymbol{A}^C = \boldsymbol{0}$). Accordingly, $\boldsymbol{v}\boldsymbol{A}^C = \boldsymbol{v}\lambda$ with $|\lambda| = 1$, $\boldsymbol{v}\boldsymbol{B}^C \neq \boldsymbol{0}$ and $\boldsymbol{v} \neq \boldsymbol{0}$ is equivalent to $\tilde{\boldsymbol{v}}\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{P}^C) = \tilde{\boldsymbol{v}}\lambda$ with $|\lambda| = 1$, $\tilde{\boldsymbol{v}} \neq \boldsymbol{0}$ $\tilde{\boldsymbol{v}}\boldsymbol{B} \neq \boldsymbol{0}$. A sufficient condition for this to hold is if $\boldsymbol{B}$ had full rank. Recall the definition of $\boldsymbol{B}$ in (88) and let $\boldsymbol{B}_{x\varepsilon}$ be a full-rank matrix, $\boldsymbol{B}$ will then be square and non-singular if $\boldsymbol{B}_{\eta,b}$ has full rank, that is if every forward-looking variable is associated with a belief shock component that is linearly independent from belief shocks associated with other forward-looking variables.[28]

When the original setup has lagged endogenous variables amongst the vector of backward-looking variables, $\boldsymbol{B}_{x\varepsilon}$ will not be square and has only full column rank, but not row rank; see also (46). In this case, unit-circle controllability depends also on unit-eigenvectors to load on non-zero rows of $\boldsymbol{B}_{x\varepsilon}$; nevertheless controllability is more likely to be assured when belief-shock loadings have full rank.

The Kalman filtering framework presented above is applicable for any shock-loadings for the endogenous forecast errors, as encoded in $\boldsymbol{B}$, irregardless of whether the projection condition is satisfied or not. The projection condition discussed above requires that the optimal Kalman gain satisfies[29]

$$\begin{bmatrix} \boldsymbol{\mathcal{G}}_{yx} & -\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_x \\ \boldsymbol{K}_y \end{bmatrix} = \boldsymbol{0} \, . \tag{95}$$

where the Kalman gain $\boldsymbol{K}$ has been partitioned into $\begin{bmatrix} \boldsymbol{K}_x' & \boldsymbol{K}_y' \end{bmatrix}'$.

Before turning to a numerical procedure that identifies shock loadings that are admissible under the projection condition, we summarize the equilibrium dynamics of the entire economy for a given set of admissible shock loadings $\boldsymbol{B}$.

[28] Note that linear independence only rules out perfect collinearity but not imperfect correlation between the belief shock components affecting different forward-looking variables.

[29] Note that $\boldsymbol{\mathcal{G}}_{yx}$ is pinned down by the full-information solution, which is certainty-equivalent and thus independent of shock variances or the measurement loadings $\boldsymbol{C}$.

3.3.2 State space of equilibrium dynamics

Given a solution for the endogenous forecast errors as in (85), equilibrium dynamics of the whole system are characterized by the evolution of central bank projections $\boldsymbol{\mathcal{S}}_{t|t}$ and projection residuals $\boldsymbol{S}^*_t$. Regarding the projections, $\boldsymbol{\mathcal{S}}'_{t|t} = \begin{bmatrix} \boldsymbol{\mathcal{X}}'_t & \boldsymbol{\mathcal{Y}}'_t \end{bmatrix}$, we need only track $\boldsymbol{\mathcal{X}}_{t|t}$ since $\boldsymbol{\mathcal{Y}}_{t|t} = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{X}}_{t|t}$. The state of the economy is described by the following vector:[30]

$$\overline{\boldsymbol{\mathcal{S}}}_t \equiv \begin{bmatrix} \boldsymbol{S}^*_{t+1} \\ \boldsymbol{\mathcal{X}}_{t+1|t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} (\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}) & \boldsymbol{0} \\ \boldsymbol{\mathcal{K}}_{\mathcal{X}}\boldsymbol{C} & \boldsymbol{\mathcal{P}} \end{bmatrix}}_{\overline{\boldsymbol{\mathcal{A}}}} \overline{\boldsymbol{\mathcal{S}}}_{t-1} + \underbrace{\begin{bmatrix} (\boldsymbol{B} - \boldsymbol{K}\boldsymbol{D}) \\ \boldsymbol{\mathcal{K}}_{\mathcal{X}}\boldsymbol{D} \end{bmatrix}}_{\overline{\boldsymbol{\mathcal{B}}}} \boldsymbol{w}_{t+1} \tag{96}$$

Recalling that $\boldsymbol{S}^*_t$ contains $\boldsymbol{X}^*_t$ and $\boldsymbol{Y}^*_t$, and $\boldsymbol{\mathcal{X}}_{t|t}$ contains $\boldsymbol{X}_{t|t}$ we can easily construct $\boldsymbol{X}_t = \boldsymbol{X}^*_t + \boldsymbol{X}_{t|t}$ and $\boldsymbol{Y}_t = \boldsymbol{Y}^*_t + \boldsymbol{\mathcal{G}}_y\boldsymbol{\mathcal{X}}_{t|t}$ and $\boldsymbol{i}_t = \boldsymbol{\mathcal{P}}_i\boldsymbol{\mathcal{X}}_{t|t}$ from the state vector $\overline{\boldsymbol{\mathcal{S}}}_t$ in (96).[31]

Notice that the transition matrix $\overline{\boldsymbol{\mathcal{A}}}$ of the joint system in (96) block-lower-triangular; its eigenvalues are thus given by the eigenvalues of its diagonal blocks $\boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}$ and $\boldsymbol{\mathcal{P}}$. As established before, both of these blocks are stable matrices; as a result, $\overline{\boldsymbol{\mathcal{A}}}$ is stable.

3.3.3 Endogenous signal: A numerical solution

This section describes a numerical algorithm that searches for shock loadings $\boldsymbol{B}_{\eta\varepsilon}$ and $\boldsymbol{B}_{\eta b}$ and that satisfy the projection condition in the endogenous signal case. That is we assume the measurement equation is characterized by

$$\boldsymbol{Z}_t = \overline{\boldsymbol{C}}_x\boldsymbol{X}_t + \boldsymbol{Y}_t \ . \tag{54}$$

By construction, we have $\boldsymbol{Z}_t = \boldsymbol{Z}_{t|t}$ and thus $\boldsymbol{Y}^*_t = -\overline{\boldsymbol{C}}_x\boldsymbol{X}^*_t$, which can be used to simplify the system a little further as follows:

[30] The projection dynamics for $\boldsymbol{\mathcal{X}}_{t|t}$ were given in (75) and we have $\tilde{\boldsymbol{Z}}_t = \boldsymbol{C}\boldsymbol{S}^*_{t-1} + \boldsymbol{D}\boldsymbol{w}_t$.

[31] Where $\boldsymbol{\mathcal{P}}_i$ denotes the top block of $\boldsymbol{\mathcal{P}}$ and thus $\boldsymbol{\mathcal{P}}_i = \begin{bmatrix} \boldsymbol{\mathcal{P}}_{ii} & \boldsymbol{\mathcal{P}}_{ix} \end{bmatrix}$ as in (65).

$$\tilde{\boldsymbol{X}}_{t+1} = \tilde{\boldsymbol{A}}\boldsymbol{X}_t^* + \tilde{\boldsymbol{B}}\boldsymbol{w}_{t+1} \tag{97}$$

$$\tilde{\boldsymbol{Z}}_{t+1} = \tilde{\boldsymbol{C}}\boldsymbol{X}_t^* + \tilde{\boldsymbol{D}}\boldsymbol{w}_{t+1} \tag{98}$$

$$\text{with} \quad \tilde{\boldsymbol{A}} = \boldsymbol{A}_{xx} - \boldsymbol{A}_{xy}\overline{\boldsymbol{C}}_x \tag{99}$$

$$\tilde{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B}_{x\varepsilon} & \boldsymbol{0} \end{bmatrix} \tag{100}$$

$$\tilde{\boldsymbol{C}} = \overline{\boldsymbol{C}}_x \left(\boldsymbol{A}_{xx} - \boldsymbol{A}_{xy}\overline{\boldsymbol{C}}_x\right) + \boldsymbol{A}_{yx} - \boldsymbol{A}_{yy}\overline{\boldsymbol{C}}_x \tag{101}$$

$$\tilde{\boldsymbol{D}} = \begin{bmatrix} (\overline{\boldsymbol{C}}_x\boldsymbol{B}_{x\varepsilon} + \boldsymbol{B}_{\eta\varepsilon}) & \boldsymbol{B}_{\eta b} \end{bmatrix} \tag{102}$$

where $\boldsymbol{A}_{xx}$, $\boldsymbol{A}_{xy}$, etc. denote suitable sub-matrices of $\boldsymbol{A}$ as known from (82).

For a given guess of $\boldsymbol{D}$, the Kalman-filtering solution to this system generates a Kalman gain $\boldsymbol{K}_x$ which can be used to form projections $\tilde{\boldsymbol{X}}_{t|t} = \boldsymbol{K}_x\tilde{\boldsymbol{Z}}_t$. What remains to be seen is whether this guess also satisfies the projection condition. The projection condition requires $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{K}_x$. Together with the projection condition, the measurement equation (54) implies $\boldsymbol{I} = \overline{\boldsymbol{C}}_x\boldsymbol{K}_x + \boldsymbol{K}_y = \left(\overline{\boldsymbol{C}}_x + \boldsymbol{\mathcal{G}}_{yx}\right)\boldsymbol{K}_x$. All told, we need to find shock loadings that support a gain $\boldsymbol{K}_x$ such that $\boldsymbol{L}\boldsymbol{K}_x = \boldsymbol{I}$ where $\boldsymbol{L} = \overline{\boldsymbol{C}}_x + \boldsymbol{\mathcal{G}}_{yx}$. We employ a numerical solver, that searches for a $\boldsymbol{D}$ which generates a Kalman gain $\boldsymbol{K}_x$ such that $\boldsymbol{L}\boldsymbol{K}_x = \boldsymbol{I}$. Given a solution for $\boldsymbol{D}$ that satisfies the projection condition $\boldsymbol{L}\boldsymbol{K}_x = \boldsymbol{I}$, we can then back out $\boldsymbol{B}_{\eta\varepsilon}$ and $\boldsymbol{B}_{\eta b}$ based on (102).

3.3.4 Exogenous signal: Analytic solution

In this section, we return to the exogenous-signal case presented in Section 3.1.2. Specifically, we derive two results: First, we establish that the projection condition does not restrict the belief shock loadings of the endogenous forecast errors when the signal is exogenous. Second, we derive an analytical expression for the restrictions on the loadings of the endogenous forecast errors on fundamental shocks resulting from the projection condition.

In order to derive an analytical solution for the endogenous forecast errors, we limit attention to the case where the entire vector of backward-looking variables is exogenous;

using notation introduced in Section 3.1.2, that means $\boldsymbol{x}_t = \boldsymbol{X}_t$.

Stated in terms of innovations, the signal extraction problem is given by the following system:

$$\tilde{\boldsymbol{X}}_t = \boldsymbol{A}_{xx}\boldsymbol{X}^*_{t-1} + \boldsymbol{B}_{x\varepsilon}\boldsymbol{\varepsilon}_t \tag{103}$$

$$\tilde{\boldsymbol{Z}}_t = \overline{\boldsymbol{C}}_x\tilde{\boldsymbol{X}}_t \tag{104}$$

where $\boldsymbol{A}_{xx} = \boldsymbol{H}_{xx}$, are appropriate sub-blocks of $\boldsymbol{H}$ and $\boldsymbol{B}_{x\varepsilon}$ as given in (49). Applying results from Section 3.3.1, a steady state Kalman filter exists with a unique gain matrix $\boldsymbol{K}_x$ and positive semi-definite $\mathrm{Var}\left(\boldsymbol{X}^*_t\right)$ provided that $(\boldsymbol{A}_{xx}, \overline{\boldsymbol{C}}_x)$ are detectable and $(\boldsymbol{A}_{xx}, \boldsymbol{B}_{x\varepsilon})$ are unit-circle controllable.[32]

Defining feature of the exogenous-signal case is that the signal extraction problem can be solved independently from the dynamics of the forward-looking variables, $\boldsymbol{Y}_t$. In particular, we have $\boldsymbol{X}^*_t = (\boldsymbol{I} - \boldsymbol{K}_x\overline{\boldsymbol{C}}_x)\tilde{\boldsymbol{X}}_t$ and can thus write

$$\tilde{\boldsymbol{X}}_t = \boldsymbol{A}_{xx}(\boldsymbol{I} - \boldsymbol{K}_x\overline{\boldsymbol{C}}_x)\tilde{\boldsymbol{X}}_{t-1} + \boldsymbol{B}_{x\varepsilon}\boldsymbol{\varepsilon}_t \tag{105}$$

Moreover, application of the projection condition also provides with a *known* Kalman gain for the forward-looking variables: $\boldsymbol{K}_y = \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{K}_x$ and the innovation dynamics of the forward-looking variables are restricted by the following transition equation:

$$\tilde{\boldsymbol{Y}}_{t+1} = \boldsymbol{A}_{yx}(\boldsymbol{I} - \boldsymbol{K}_x\overline{\boldsymbol{C}}_x)\tilde{\boldsymbol{X}}_{t-1} + \boldsymbol{A}_{yy}(\boldsymbol{I} - \boldsymbol{K}_y\overline{\boldsymbol{C}}_x)\tilde{\boldsymbol{Y}}_{t-1} + \boldsymbol{\eta}_{t+1} \tag{106}$$

where the endogenous forecast errors, $\boldsymbol{\eta}_t$, remain to be derived. As before, we seek $\boldsymbol{\eta}_t = \boldsymbol{B}_{\eta\varepsilon}\,\boldsymbol{\varepsilon}_t + \boldsymbol{B}_{\eta b}\,\boldsymbol{b}_t$, with loadings $\boldsymbol{B}_{\eta\varepsilon}$ and $\boldsymbol{B}_{\eta b}\,\boldsymbol{b}_t$ that satisfy the projection condition. In addition, in order to ensure stable dynamics of $\tilde{\boldsymbol{Y}}_t$, $\boldsymbol{A}_{yy}(\boldsymbol{I} - \boldsymbol{K}_y\overline{\boldsymbol{C}}_x)$ has to be a stable

[32] Analogously to derivation shown in Section 3.3.1, application of Theorem 1 requires to transform (103) and (103) into "ABCD" form and establish detectability and unit-circle controllability with respect to matrices derived from the transformed system. However, those conditions turn out to be identical to requiring detectability of $(\boldsymbol{A}_{xx}, \overline{\boldsymbol{C}}_x)$ and unit-circle controllability of $(\boldsymbol{A}_{xx}, \boldsymbol{B}_{x\varepsilon})$.

matrix.[33]

The projection condition requires $\text{Cov}\,(\tilde{\boldsymbol{Y}}_t, \tilde{\boldsymbol{Z}}_t) = \boldsymbol{\mathcal{G}}_{yx}\,\text{Cov}\,(\tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{Z}}_t)$. Due to the exogeneity of $\tilde{\boldsymbol{Z}}_t$ — and thus $\text{Cov}\,(\boldsymbol{b}_t, \tilde{\boldsymbol{Z}}_{\boldsymbol{t}}) = 0$ — this covariance restriction does not affect admissible belief shock loadings $\boldsymbol{B}_{\eta b}$.

PROPOSITION 1 (Unrestricted belief-shock loadings when the signal is exogenous). *When the signal is exogenous, as given by* (103) *and* (104)*, there are no restrictions on* $\boldsymbol{B}_{\eta b}$ *for an equilibrium as defined above to exist.*

Proof. Formally, we can decompose $\tilde{\boldsymbol{Y}}_t$ into two pieces: a component that reflects the history of fundamental shocks $\boldsymbol{\varepsilon}^t$ and a component driven by belief shocks.

$$\tilde{\boldsymbol{Y}}_{t+1} = \tilde{\boldsymbol{Y}}^{\varepsilon}_{t+1} + \tilde{\boldsymbol{Y}}^{b}_{t+1} \tag{107}$$

$$\tilde{\boldsymbol{Y}}^{\varepsilon}_{t+1} \equiv \boldsymbol{A}_{yx}(\boldsymbol{I} - \boldsymbol{K}_x\overline{\boldsymbol{C}}_x)\tilde{\boldsymbol{X}}_{t-1} + \boldsymbol{A}_{yy}(\boldsymbol{I} - \boldsymbol{K}_y\overline{\boldsymbol{C}}_x)\tilde{\boldsymbol{Y}}^{\varepsilon}_{t-1} + \boldsymbol{B}_{\eta\varepsilon}\boldsymbol{\varepsilon}_{t+1} \tag{108}$$

$$\tilde{\boldsymbol{Y}}^{b}_{t+1} \equiv \boldsymbol{A}_{yy}(\boldsymbol{I} - \boldsymbol{K}_y\overline{\boldsymbol{C}}_x)\tilde{\boldsymbol{Y}}^{b}_{t-1} + \boldsymbol{B}_{\eta b}\boldsymbol{b}_{t+1} \tag{109}$$

and since, for any h, we have $E(\boldsymbol{b}_{t+h}|\tilde{\boldsymbol{Z}}_t) = \boldsymbol{0}$ it follows have $E(\tilde{\boldsymbol{Y}}_t|\tilde{\boldsymbol{Z}}_t) = \boldsymbol{0}$ for any $\boldsymbol{B}_{\eta b}$. (Further requirements for the existence of an equilibrium as described in Definition 1 are the stability of $\boldsymbol{A}_{yy}(\boldsymbol{I} - \boldsymbol{K}_y\overline{\boldsymbol{C}}_x)$ and fundamental shock loadings $\boldsymbol{B}_{\eta\varepsilon}$ that satisfy the projection condition. However, satisfaction of those conditions does not depend on $\boldsymbol{B}_{\eta b}$.) ∎

Without proof, notice that the result also goes through, when part of the backward-looking variables were endogenous (while the signal remains exogenous).

Finally, we can also state simple expressions to construct fundamental shock loadings $\boldsymbol{B}_{\eta\varepsilon}$ that satisfy the projection condition. Notice that the projection condition requires $\text{Cov}\,(\tilde{\boldsymbol{W}}_t, \tilde{\boldsymbol{Z}}_t) = \boldsymbol{0}$ where $\tilde{\boldsymbol{W}}_t \equiv \tilde{\boldsymbol{Y}}^{\varepsilon}_t - \boldsymbol{\mathcal{G}}_{yx}\tilde{\boldsymbol{X}}_t$. (In light of Proposition 1, we can neglect the effects of belief shocks and $\tilde{\boldsymbol{W}}_t$ has been defined with reference to $\tilde{\boldsymbol{Y}}^{\varepsilon}_t$, as defined in the proof to Proposition 1.)

[33] Please note that existence of a steady-state Kalman filter for (103) and (104) assures stable dynamics of $\tilde{\boldsymbol{X}}_t$.

Let $\boldsymbol{\Sigma}_{wx} \equiv \operatorname{Cov}(\tilde{\boldsymbol{W}}_t, \tilde{\boldsymbol{X}}_t)$. Based on (103) and (106), and with $\boldsymbol{A}_{wx} = \boldsymbol{A}_{yx} - \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{A}_{xx}$, the dynamics of $\tilde{\boldsymbol{W}}_t$ are given by

$$\tilde{\boldsymbol{W}}_{t+1} = \boldsymbol{A}_{wx}(\boldsymbol{I} - \boldsymbol{K}_x\overline{\boldsymbol{C}}_x)\tilde{\boldsymbol{X}}_{t-1} + \boldsymbol{A}_{yy}(\boldsymbol{I} - \boldsymbol{K}_y\overline{\boldsymbol{C}}_x)\tilde{\boldsymbol{W}}_{t-1} + (\boldsymbol{B}_{\eta\varepsilon} - \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{B}_{x\varepsilon})\boldsymbol{\varepsilon}_{t+1} \tag{110}$$

$$\begin{aligned}\boldsymbol{\Sigma}_{wx} = &\left\{\boldsymbol{A}_{wx}(\boldsymbol{I} - \boldsymbol{K}_x\overline{\boldsymbol{C}}_x)\boldsymbol{\Sigma}_{xx} + \boldsymbol{A}_{yy}(\boldsymbol{I} - \boldsymbol{K}_y\overline{\boldsymbol{C}}_x)\boldsymbol{\Sigma}_{wx}\right\}(\boldsymbol{I} - \boldsymbol{K}_x\overline{\boldsymbol{C}}_x)'\boldsymbol{A}_{xx}' \\ &+ (\boldsymbol{B}_{\eta\varepsilon} - \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{B}_{x\varepsilon})\boldsymbol{B}_{x\varepsilon}' \end{aligned} \tag{111}$$

where $\boldsymbol{\Sigma}_{xx} = \operatorname{Var}(\tilde{\boldsymbol{X}}_t)$ is known from solving the steady-state Kalman filter for the exogenous signal case. The only unknowns in (111) are $\boldsymbol{\Sigma}_{wx}$ and $\boldsymbol{B}_{\eta\varepsilon}$ and we seek to find $\boldsymbol{B}_{\eta\varepsilon}$ such that $\boldsymbol{\Sigma}_{wx}\overline{\boldsymbol{C}}_x' = \boldsymbol{0}$.

Valid values of $\boldsymbol{\Sigma}_{wx}$ must lie in the nullspace of $\overline{\boldsymbol{C}}_x'$. Specifically, given a $(N_x - N_z) \times N_x$ matrix $\boldsymbol{N}$ such that $\boldsymbol{N}\overline{\boldsymbol{C}}' = \boldsymbol{0}$.[34] we can construct valid candidates for $\boldsymbol{\Sigma}_{wx}$ by choosing an arbitrary $N_y \times (N_x - N_z)$ matrix $\boldsymbol{G}$ and let $\boldsymbol{\Sigma}_{wx} = \boldsymbol{G}\boldsymbol{N}$.

For a given candidate $\boldsymbol{\Sigma}_{wx} = \boldsymbol{G}\boldsymbol{N}$, $\boldsymbol{B}_{\eta\varepsilon}$ must thus satsfy the following condition:

$$\boldsymbol{B}_{\eta\varepsilon}\boldsymbol{B}_{x\varepsilon}' = \boldsymbol{f}(\boldsymbol{G}) \tag{112}$$

$$\begin{aligned}\text{where}\quad \boldsymbol{f}(\boldsymbol{G}) \equiv\ &\boldsymbol{G}\boldsymbol{N} + \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{B}_{x\varepsilon}\boldsymbol{B}_{x\varepsilon}' - (\boldsymbol{A}_{wx}(\boldsymbol{I} - \boldsymbol{K}_x\overline{\boldsymbol{C}}_x)\boldsymbol{\Sigma}_{xx}(\boldsymbol{I} - \boldsymbol{K}_x\overline{\boldsymbol{C}}_x)'\boldsymbol{A}_{xx}' \\ &- \boldsymbol{A}_{yy}(\boldsymbol{I} - \boldsymbol{K}_y\overline{\boldsymbol{C}}_x)\boldsymbol{G}\boldsymbol{N}(\boldsymbol{I} - \boldsymbol{K}_x\overline{\boldsymbol{C}}_x)'\boldsymbol{A}_{xx}'\end{aligned} \tag{113}$$

The ability to solve (112) for $\boldsymbol{B}_{\eta\varepsilon}$ depends on the dimension of the problem. When $N_x = N_\varepsilon$, the number of exogenous variables is identical to the number of exogenous shocks, and $\boldsymbol{B}_{x\varepsilon}$ is invertible. When $|\boldsymbol{B}_{x\varepsilon}| \neq 0$ it is straightforward to solve (112) for $\boldsymbol{B}_{\eta\varepsilon}$ given an arbitrary choice of $\boldsymbol{G}$:

$$\boldsymbol{B}_{\eta\varepsilon} = \boldsymbol{f}(\boldsymbol{G})(\boldsymbol{B}_{x\varepsilon}')^{-1}\ . \tag{114}$$

[34] A matrix $\boldsymbol{N}$ such that $\boldsymbol{N}\overline{\boldsymbol{C}}' = \boldsymbol{0}$ can readily be obtained from the SVD decomposition of $\overline{\boldsymbol{C}} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}'$ where $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthonormal, $\boldsymbol{S} = \begin{bmatrix}\boldsymbol{S}_1 & \boldsymbol{0}\end{bmatrix}$ and $\boldsymbol{S}_1$ is a $N_z \times N_z$ diagonal matrix. Partition $\boldsymbol{V}$ conformably into $\boldsymbol{V} = \begin{bmatrix}\boldsymbol{V}_1 & \boldsymbol{V}_2\end{bmatrix}$ such that $\overline{\boldsymbol{C}} = \boldsymbol{U}\boldsymbol{S}_1\boldsymbol{V}_1'$. Since $\boldsymbol{V}$ is orthonormal we have $\boldsymbol{V}_2'\boldsymbol{V}_1 = \boldsymbol{0}$. Choosing $\boldsymbol{N} = \boldsymbol{V}_2'$ then ensures $\boldsymbol{N}\overline{\boldsymbol{C}}' = \boldsymbol{0}$.

In case of $N_x > N_\varepsilon$, there is not necessarily a $\boldsymbol{B}_{\eta\varepsilon}$ that solves (112) for any $\boldsymbol{G}$. Instead, $\boldsymbol{G}$ needs to be chosen such that $\boldsymbol{f}(\boldsymbol{G})\left(\boldsymbol{I} - \boldsymbol{B}_{x\varepsilon}(\boldsymbol{B}'_{x\varepsilon}\boldsymbol{B}_{x\varepsilon})^{-1}\boldsymbol{B}'_{x\varepsilon}\right) = \boldsymbol{0}$, which can be obtained numerically.[35] For such a choice of $\boldsymbol{G}$, a valid $\boldsymbol{B}_{\eta\varepsilon}$ is given by $\boldsymbol{B}_{\eta\varepsilon} = \boldsymbol{f}(\boldsymbol{G})\boldsymbol{B}_{x\varepsilon}(\boldsymbol{B}'_{x\varepsilon}\boldsymbol{B}_{x\varepsilon})^{-1}$.

3.4 Variance Bound in the General Case

In our description of the Fisher-economy example in Section 2, we have highlighted the existence of an upper bound on the variance of endogenous variables (inflation in that example) that holds across all of the equilibria considered. Here, we show how these arguments can be extended to the general case.

Specifically, analogously to the example from Section 2, we consider the following case of an endogenous signal, where the measurement vector conveys a noisy signal about the vector of forward-looking variables:

$$\boldsymbol{Z}_t = \boldsymbol{Y}_t + \boldsymbol{\nu}_t \qquad\qquad \boldsymbol{\nu}_t \sim N(\boldsymbol{0}, \boldsymbol{\Omega}_{\nu\nu}) \tag{115}$$

where $\boldsymbol{\nu}_t$ is a vector of *iid* measurement errors. In terms of our general framework, laid out above, these measurement errors would usually be tracked as part of the vector of backward-looking variables, $\boldsymbol{X}_t$, and (115) can be recognized as a special case of (54). To facilitate the derivation of the variance bound, partition the vector of backward-looking variables into $\boldsymbol{X}_t = \begin{bmatrix} \boldsymbol{x}'_t & \boldsymbol{\nu}'_t \end{bmatrix}'$ where $\boldsymbol{x}_t$ denotes the backward-looking variables present in the full-information version of a given model. As the measurement errors have no role in the full-information version of the model, the projection condition reduces to $\boldsymbol{Y}_{t|t} = \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{x}_{t|t}$.[36]

Furthermore, as in the Fisher-economy example of Section 2, consider the case where the backward-looking variables are purely exogenous, that is $\boldsymbol{x}_t = \boldsymbol{H}_{xx}\boldsymbol{x}_{t-1} + \boldsymbol{B}_{x\varepsilon}\boldsymbol{\varepsilon}_t$, $\boldsymbol{\varepsilon}_t \sim N(\boldsymbol{0}, \boldsymbol{I})$, and $\text{Var}(\boldsymbol{x}_t)$ is given independently from any particular equilibrium for the endogenous variables.[37]

[35]Note that, as introduced in Section 3.1, $\boldsymbol{B}_{x\varepsilon}$ has full-column rank which ensures that $|\boldsymbol{B}'_{x\varepsilon}\boldsymbol{B}_{x\varepsilon}| \neq 0$.

[36]In a slight abuse of notation, we continue to use $\boldsymbol{\mathcal{G}}_{yx}$ — as introduced in equation (64) above — when referring to the coefficients mapping $\boldsymbol{x}_t$ — rather than $\boldsymbol{X}_t$ — into the vector of endogenous variables.

[37]Given values for $\boldsymbol{H}_{xx}$ and $\boldsymbol{B}_{x\varepsilon}$, and requiring that $\boldsymbol{H}_{xx}$ be stable, $\text{Var}(\boldsymbol{x}_t)$ is given by the solution to a

By construction, we have $\boldsymbol{Z}_{t|t} = \boldsymbol{Z}_t$ and can thus deduce that $\boldsymbol{Y}^*_t = -\boldsymbol{\nu}^*_t$.[38] Together with the projection condition (80) and the law of total variance, we then obtain:

$$\boldsymbol{Y}_t = \boldsymbol{\mathcal{G}}_{yx}\, \boldsymbol{x}_{t|t} - \boldsymbol{\nu}^*_t \tag{116}$$

$$\Rightarrow \quad \text{Var}\left(\boldsymbol{Y}_t\right) = \boldsymbol{\mathcal{G}}_{yx}\, \text{Var}\left(\boldsymbol{x}_{t|t}\right) \boldsymbol{\mathcal{G}}'_{yx} + \text{Var}\left(\boldsymbol{\nu}^*_t\right) \tag{117}$$

$$\Rightarrow \quad \text{Var}\left(\boldsymbol{Y}_t\right) \leq \boldsymbol{\mathcal{G}}_{yx}\, \text{Var}\left(\boldsymbol{x}_t\right) \boldsymbol{\mathcal{G}}'_{yx} + \text{Var}\left(\boldsymbol{\nu}_t\right) \tag{118}$$

where the weak inequality is understood as indicating a semi-definite difference between matrices. The absence of covariance terms in (117) follows from the optimality of projections, which requires projection residuals, like $\boldsymbol{\nu}^*_t$ to be orthogonal to $\boldsymbol{Z}^t$ or any functions thereof (like $\boldsymbol{x}_{t|t}$). In addition, the law of total variance implies $\text{Var}\left(\boldsymbol{x}_t\right) = \text{Var}\left(\boldsymbol{x}_{t|t}\right) + \text{Var}\left(\boldsymbol{x}_t | \boldsymbol{Z}^t\right) \geq \text{Var}\left(\boldsymbol{x}_{t|t}\right)$ and an analogous expression for $\text{Var}\left(\boldsymbol{\nu}_t\right)$.

However, in contrast to the simple Fisher example, where $\pi_t = \bar{g} r_t - \nu_t$ was also a valid equilibrium, its analogue in the general case — given by $\boldsymbol{Y}^B_t \equiv \boldsymbol{\mathcal{G}}_{yx} \boldsymbol{x}_t - \boldsymbol{\nu}_t$ — will typically not be a possible outcome in equilibrium. Nevertheless, (118) provides an upper bound on the variability of equilibria for $\boldsymbol{Y}_t$ across all potential equilibria in our environment. To demonstrate why $\boldsymbol{Y}^B_t$ is generally not an equilibrium outcome, we consider the following, simplified version of the general setup known from (49), together with the signal given by (115):

$$\boldsymbol{x}_{t+1} = \boldsymbol{H}_{xx} \boldsymbol{x}_t + \boldsymbol{B}_{x\varepsilon} \boldsymbol{\varepsilon}_{t+1} \tag{119}$$

$$E_t \boldsymbol{Y}_{t+1} = \boldsymbol{H}_{yx} \boldsymbol{x}_t + \boldsymbol{H}_{yx} \boldsymbol{Y}_t + \hat{\boldsymbol{H}}_{yx} \boldsymbol{x}_{t|t} + \hat{\boldsymbol{H}}_{yx} \boldsymbol{Y}_{t|t} \tag{120}$$

$$\boldsymbol{Z}_t = \boldsymbol{Y}_t + \boldsymbol{\nu}_t, \quad \boldsymbol{\nu}_t \sim N(\boldsymbol{0}, \boldsymbol{\Omega}_{\nu\nu}) \tag{115}$$

We proceed by showing that evaluation of (115) using the guess $\boldsymbol{Y}^B_t \equiv \boldsymbol{\mathcal{G}}_{yx} \boldsymbol{x}_t - \boldsymbol{\nu}_t$ yields a contradiction: The left-hand side of (115) becomes a function of $\boldsymbol{x}_t$ alone, while the right

Lyapunov equation, $\text{Var}\left(\boldsymbol{x}_t\right) = \boldsymbol{H}_{xx}\, \text{Var}\left(\boldsymbol{x}_t\right) \boldsymbol{H}'_{xx} + \boldsymbol{B}_{x\varepsilon} \boldsymbol{B}'_{x\varepsilon}$, which can be obtained using standard methods (Sargent and Ljungqvist, 2004; Hamilton, 1994).

[38] Asterisks continue to denote projection residuals $\boldsymbol{Y}^*_t \equiv \boldsymbol{Y}_t - \boldsymbol{Y}_{t|t}$ and $\boldsymbol{\nu}^*_t \equiv \boldsymbol{\nu}_t - \boldsymbol{\nu}_{t|t}$.

hand-side becomes a function of $\boldsymbol{x}_t$, $\boldsymbol{x}_{t|t}$ and $\boldsymbol{\nu}_t$:

$$\boldsymbol{\mathcal{G}}_{yx}\boldsymbol{H}_{xx}\,\boldsymbol{x}_t \stackrel{?}{=} \left(\boldsymbol{H}_{yx} + \boldsymbol{H}_{yx}\boldsymbol{\mathcal{G}}_{yx}\right)\boldsymbol{x}_t + \left(\hat{\boldsymbol{H}}_{yx} + \hat{\boldsymbol{H}}_{yx}\boldsymbol{\mathcal{G}}_{yx}\right)\boldsymbol{x}_{t|t} - \hat{\boldsymbol{H}}_{yy}\boldsymbol{\nu}_t \tag{121}$$

However, together with (115), the candidate $\boldsymbol{Y}_t^B$ implies that the signal becomes a function of $\boldsymbol{x}_t$ alone, $\boldsymbol{Z}_t = \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{x}_t$, so that the projections $\boldsymbol{x}_{t|t}$ are independent from ν_t.[39] A necessary condition for (121) to hold would thus be to require that $\boldsymbol{H}_{yy} = \boldsymbol{0}$, which was indeed the case in the simple Fisher example.[40] However, in general, $\boldsymbol{H}_{yy}$ is not zero, as illustrated, for example, in the case of the New Keynesian model analyzed in Section 4.

3.5 Determinate outcomes from a different rule

So far, we considered only reaction functions for the policy instrument that, as in (50), respond to optimal projections of backward- and forward-looking variables. Rules of this form could, for example, be motivated by noting that a given rule was deemed desirable under full information and pointing to a certainty equivalence argument. Indeed, based on reasoning along those lines, Svensson and Woodford (2004) derive optimal reactions in a form consistent with (50). A central message of this paper, is to note that the interaction of the policymaker's filtering and private-sector agents' forward looking behavior, embodied by the linear difference system in (49), lead to a multiplicity of equilibria that is general inherent in the class of models studied here. In particular, rules of the form in (50) commit the policymaker only in her responses to *projected* input variables, but not in her responses to *incoming data*. Responses to incoming data are rather governed by the policymakers filtering efforts, which depend on equilibrium outcomes, and — sadly for the purpose of achieving uniqueness — end up stabilizing many possible equilibria.

This section describes an alternative class of policy rules, that satisfies the same (if

[39] Notice that in (121), we also imposed the projection condition on the candidate outcome $\boldsymbol{Y}_{t|t}^B = \boldsymbol{\mathcal{G}}_{yx}\boldsymbol{x}_{t|t}$ as required in equilibrium. Even in the absence of the equilibrium condition holding, projections $\boldsymbol{Y}_{t|t}^B$ would, however, be a function of $\boldsymbol{x}^t$ alone and independent of $\boldsymbol{\nu}_t$.

[40] In the Fisher example, (120) collapses to a combination of the Fisher equation and the Taylor rule, $E_t\pi_{t+1} = \phi\pi_{t|t} - r_t$, which does not feature the current value of the forward-looking variable π_t.

not easier) informational requirements as (50), while also achieving unique equilibrium determinacy. An example of such a rule has already been described in the context of our Fisher-economy example in Section 2. In this case, the reaction function of the policy instrument responds directly to incoming data, as captured by Z_t (and possibly also lags thereof) instead of projections, which are an endogenously determined function of Z^t. Specifically, we generalize the specific example from Section 2, to an environment that maintains the following features:

1. In the full-information case, the policy rule responds only to forward-looking variables.

$$\boldsymbol{i}_t = \boldsymbol{\Phi}_i \boldsymbol{i}_{t-1} + \boldsymbol{\Phi}_y \boldsymbol{Y}_t \tag{122}$$

   An example of such a rule are outcome-based Taylor rules, while policy rules with stochastic intercept are excluded.

2. Forward-looking behavior of the private sector is characterized by an expectational difference system similar to (49) except that — for simplicity — central-bank projections are assumed to enter only via the policy rule, that is:

$$\boldsymbol{J} E_t \boldsymbol{S}_{t+1} = \boldsymbol{H}\boldsymbol{S}_t + \boldsymbol{H}_i \boldsymbol{i}_t \qquad |\boldsymbol{J}| \neq 0 \tag{123}$$

   where $\boldsymbol{S}_t$ continues to denote the stacked vector of backwar- and looking variables.

3. As before, we assume the values of the policy-rule coefficients $\boldsymbol{\Phi}_i$ and $\boldsymbol{\Phi}_y$ to be such that, when the reaction function (122) is combined with the difference system in (123), there is a unique full-information rational expectations equilibrium.

4. As in our discussion of the variance bounds in the general case, the measurement vector is supposed to convey a noisy signal of every forward-looking variable.

$$\boldsymbol{Z}_t = \boldsymbol{Y}_t + \boldsymbol{\nu}_t \qquad \boldsymbol{\nu}_t \sim N(\boldsymbol{0}, \boldsymbol{\Omega}_{\nu\nu}) \tag{115}$$

For simplicity, we continue to assume that ν_t is serially uncorrelated, though equilibrium uniqueness will not depend on this property.

When the policymaker can only observe the history of $\boldsymbol{Z}_t$ instead of $\boldsymbol{Y}_t$ and $\boldsymbol{X}_t$, the rule (122) cannot be implemented by the policymaker. Instead of evaluating the rule using optimal projections of $\boldsymbol{Y}_t$ onto $\boldsymbol{Z}^t$, a policymaker could also consider to simply replace $\boldsymbol{Y}_t$ by its noisy signal. That is, the policymaker could set the policy instrument according to

$$\boldsymbol{i}_t = \boldsymbol{\Phi}_i \boldsymbol{i}_{t-1} + \boldsymbol{\Phi}_y \left(\boldsymbol{Y}_t + \boldsymbol{\nu}_t \right) \ . \tag{124}$$

The economy is then described by the expectational difference system (123) and the policy rule (124). Importantly, equilibrium does not hinge on any signal extraction efforts and its determination can be studied using the standard methods — developed, among others, by Blanchard and Kahn (1980), King and Watson (1998), Klein (2000) and Sims (2002) — described above in the context of the full-information case. In particular, notice that the full-information system described by (123) and (122) differs only from the "signal-rule system", described by (123) and (124), only in the presence of additional, exogenous driving variables given by $\boldsymbol{\nu}_t$, while sharing an identical transmission of endogenous variables. Since we assumed that the the full-information system given by (123) and (122) satisfies the conditions for a unique equilibrium, it follows directly that the "signal-rule system", described by (123) and (124), also generates a unique equilibrium. In particular, using notation introduced above in our characterization of full-information outcomes for the general case, equilibrium outcomes have the following form:

$$\boldsymbol{Y}_t = \boldsymbol{\mathcal{G}}_{yx} \boldsymbol{X}_t + \boldsymbol{\mathcal{G}}_{yi} \boldsymbol{i}_{t-1} + \boldsymbol{\mathcal{G}}_{y\nu} \boldsymbol{\nu}_t \tag{125}$$

$$E_t \boldsymbol{X}_{t+1} = \boldsymbol{\mathcal{P}}_{xx} \boldsymbol{X}_t + \boldsymbol{\mathcal{P}}_{yi} \boldsymbol{i}_{t-1} + \boldsymbol{\mathcal{P}}_{x\nu} \boldsymbol{\nu}_t \tag{126}$$

where $\boldsymbol{\mathcal{G}}_{yx}$, $\boldsymbol{\mathcal{G}}_{yi}$, $\boldsymbol{\mathcal{P}}_{xx}$, $\boldsymbol{\mathcal{P}}_{yi}$ take identical values as in the full-information solution.[41]

[41] Alternatively stated, the full-information outcomes are identical to (125) and (126), with identical coefficient values, except for $\boldsymbol{\mathcal{G}}_{y\nu}$ and $\boldsymbol{\mathcal{P}}_{x\nu}$ being equal to zero.

While the ability to achieve equilibrium uniqueness might seem, at least initially. We should caution, that the dependence of endogenous outcomes on signal noise in (125) and (126) can lead to potentially highly undesirable fluctuations caused by measurement noise. Effectively, while maintaining the requirement that policy can only respond to observables spanned by $\boldsymbol{Z}^t$, determinacy is achieved under the "signals rule" by committing the policy rule respond to incoming noise with the same sensitivity as it does to $\boldsymbol{Y}_t$. In particular, in the context of the Fisher-example described in Section 2, we had the case where $\boldsymbol{\mathcal{G}}_{y\nu} = \mathbf{0}$ and $\boldsymbol{X}_t$ was purely exogenous. With this particular configuration, the variance bounds established above indicates that *any* admissible equilibrium under the corresponding projections-based policy rule, that is $\boldsymbol{i}_t = \boldsymbol{\Phi}_i \boldsymbol{i}_{t-1} + \boldsymbol{\Phi}_y \boldsymbol{Y}_{t|t}$, generates less-variable outcomes.[42]

Our argument could even be extended beyond the setting laid out above — for example by replacing policy rule responses to backward-looking variables or expectations of future realizations of forward-looking variables by noisy signals of those without altering our conclusions (albeit at the cost of further notational complexity).

[42] Specifically, with $\mathrm{Var}\,(\boldsymbol{Y}_t) \leq \boldsymbol{\mathcal{G}}_{yx}\,\mathrm{Var}\,(\boldsymbol{X}_t)\boldsymbol{\mathcal{G}}'_{yx} + \mathrm{Var}\,(\boldsymbol{\nu}_t)$ the difference between the variance-covariance matrix of outcomes for the forward-looking variables under the projection-based rule and its counterpart generated by the signal-based rule is positive semi-definite; so that any quadratic loss function over $\boldsymbol{Y}_t$ would at least weakly prefer outcomes under the projections-based rule.

4 A New Keynesian Model

Next we turn to the numerical solution of a 3 equation New Keynesian model. The model consists of a household that has the same information set as households in the full information version of the model as well as a central bank that only observes noisy measurements of inflation and the level of real GDP. Throughout this section we assume that the central bank follows a monetary policy rule in which it reacts to (Among other variables) its best estimate of inflation and the output *gap*. Thus the central bank not only has to infer the true level of output from its noisy signal, but it faces a possibly even more difficult signal extraction problem because it has to infer the best estimate of the output *gap* the available data. In our version of the New Keynesian models, we posit that the log difference in potential GDP $\bar{y}_t$ is a stationary AR(1) process, which makes the real rate r_t in this economy a linear function of the expected log difference in potential output. We also introduce a cost-push shock u_t as well as the two measurement errors in inflation and the level of output. To calibrate the model, we have chosen standard parameters in the literature where possible. To calibrate the standard deviation of the iid measurement error process, we have relied on estimates from Lubik & Matthes (2016). In that paper the authors estimate measurement error processes for inflation and real GDP growth that are AR(1) processes. For inflation, we choose the standard deviation of our measurement error process to math the unconditional standard deviation for the inflation measurement error process in Lubik & Matthes (2016). Since the measurement error in inflation is estimated to be only mildly autocorrelated in that paper (with a point estimate of around 0.1 for the AR coefficient) the switch from an autoregressive measurement error process to an iid one seems innocuous. For real GDP, we assume that the level of (log) GDP is measured with iid error. Note that this automatically induces autocorrelation in the measurement error for the log difference of GDP, which is what Lubik & Matthes (2016) find. In that case the standard deviation of the measurement error in the log difference will be twice the standard deviation of the measurement error in levels, which we exploit in our calibration. We match the standard deviation of our iid measurement error to half of the unconditional standard deviation of the measurement error for GDP growth found in Lubik & Matthes (2016). Standard deviations of all shocks are expressed in annualized percentages.

Below we give the equations for our model (except for the filtering equations, which we do not repeat here for the sake of brevity) and a table with the exact numerical values of parameters. The model is reasonably standard, but nonetheless some elements are worth pointing out: The model includes a New Keynesian Phillips curve with a backward-looking component for inflation $\gamma\pi_{t-1}$. The central bank follows a policy rule that reacts to filtered estimates of inflation and the output gap x_t. The frictionless real rate r_t is a linear function of expected growth in potential real GDP. Finally, the level of GDP in this economy is by construction equal to the growth rate in potential GDP plus the sum of lagged potential GDP and the current output gap.

$$(1-\gamma\beta)\pi_t = \gamma\pi_{t-1} + \beta E_t\pi_{t+1} + \kappa x_t \tag{127}$$

$$i_t = r_t + E_t\pi_{t+1} + \sigma(E_t x_{t+1} - x_t) \tag{128}$$

$$i_t = \phi_\pi \pi_{t|t} + \phi_x x_{t|t} \tag{129}$$

$$r_t = \sigma E_t \Delta\bar{y}_{t+1} \tag{130}$$

$$\Delta\bar{y}_t = \rho_y \Delta\bar{y}_{t-1} + \varepsilon_t^y \qquad \varepsilon_t^y \sim N(0, \sigma_y^2) \tag{131}$$

$$u_t = \rho_u u_{t-1} + \varepsilon_t^u \qquad \varepsilon_t^u \sim N(0, \sigma_u^2) \tag{132}$$

$$Z_t = \begin{bmatrix} \pi_t + \nu_t^\pi \\ \Delta\bar{y}_t + \bar{y}_{t-1} + \nu_t^y + x_t \end{bmatrix} \qquad \begin{bmatrix} \nu_t^\pi \\ \nu_t^x \end{bmatrix} \sim N\left(\mathbf{0}, \begin{bmatrix} \sigma_\pi^2 & 0 \\ 0 & \sigma_x^2 \end{bmatrix}\right) \tag{133}$$

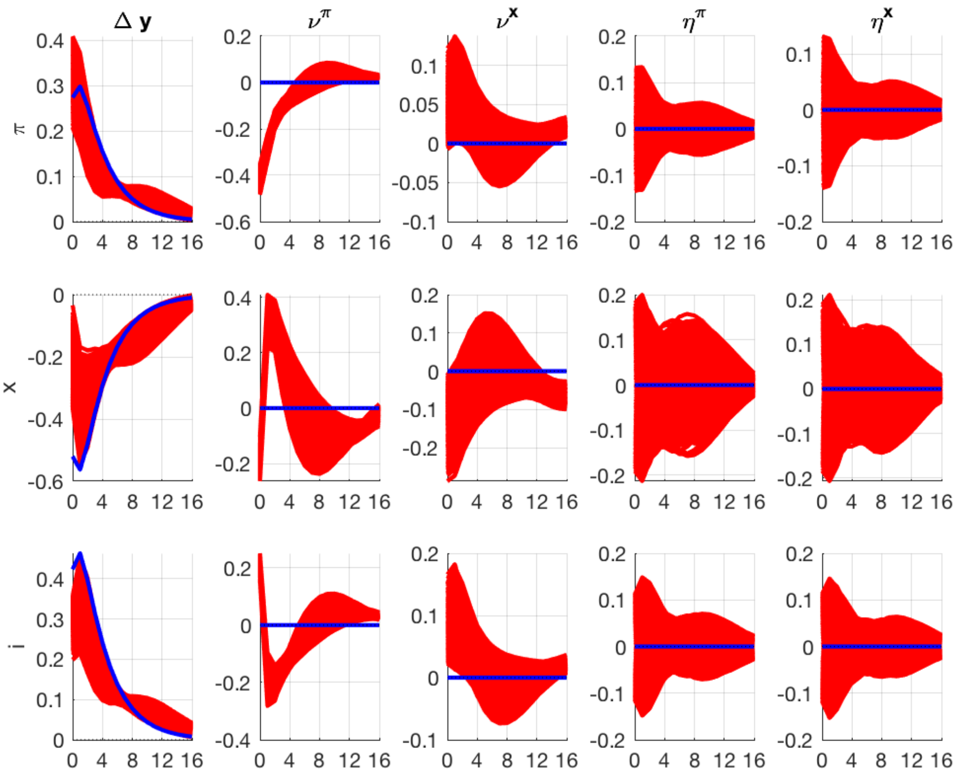

Figure 1: Impulse responses for the New Keynesian model under full information (blue) as well as various limited information equilibria. Each row represents the response of a specific variable to the shocks in the model whereas each column represent the responses of the endogenous variables to a specific shock.

Table 1: Parameters for NK model

Parameter	Value
β	0.99
σ	1.00
ϕ	1.00
γ	0.25
θ	0.75
ϕ_π	2.50
ϕ_x	0.50
ρ_y	0.75
σ_y	0.30
σ_π	0.80
σ_x	1.39
κ	$(1-\theta)(1-\beta\theta)/\theta(\sigma+\phi) = 0.17$

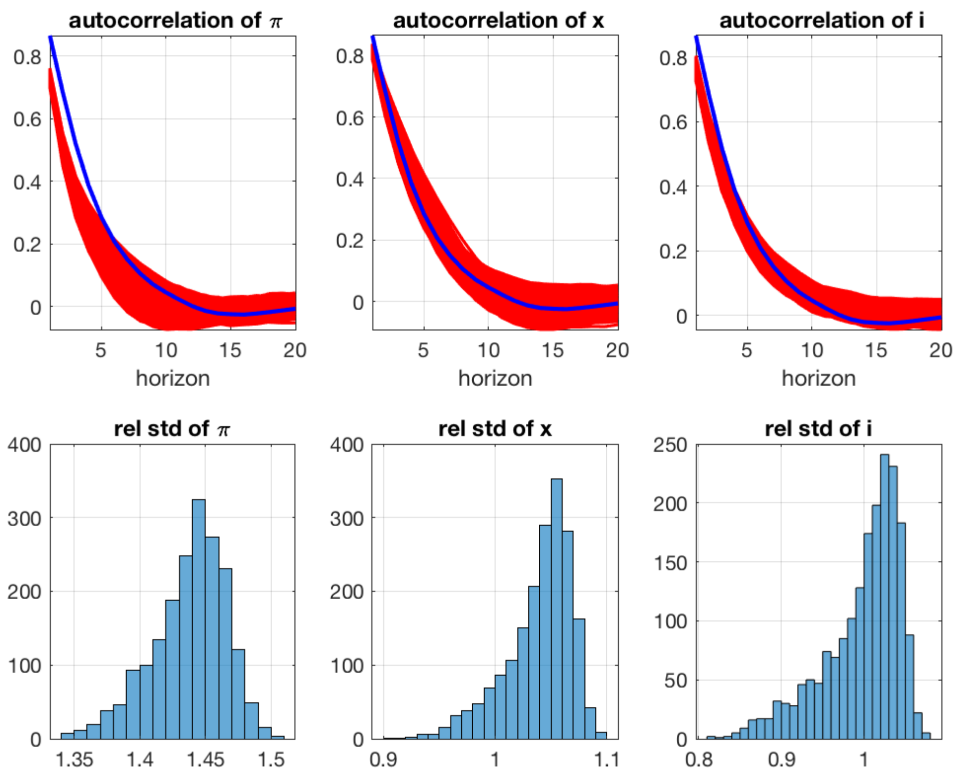

Figure 2: Moments of endogenous variables for the New Keynesian model under full information (blue) as well as various limited information equilibria.

5 Conclusion

[TO BE WRITTEN]

References

Anderson, B. D. O. and J. B. Moore (1979). *Optimal Filtering.* Information and System Sciences Series. Englewood Cliffs, New Jersey: Prentice-Hall Inc.

Anderson, E. W., E. R. McGrattan, L. P. Hansen, and T. J. Sargent (1996). Mechanics of forming and estimating dynamic linear economies. In H. M. Amman, D. A. Kendrick, and J. Rust (Eds.), *Handbook of Computational Economics*, Volume 1, Chapter 4, pp. 171–252. Elsevier.

Aoki, K. (2006, January). Optimal commitment policy under noisy information. *Journal of Economic Dynamics and Control 30*(1), 81–109.

Baxter, B., L. Graham, and S. Wright (2011, March). Invertible and non-invertible information sets in linear rational expectations models. *Journal of Economic Dynamics and Control 35*(3), 295–311.

Blanchard, O. J. and C. M. Kahn (1980, July). The solution of linear difference models under rational expectations. *Econometrica 48*(5), 1305–1312.

Evans, G. W. and B. McGough (2005, April). Stable sunspot solutions in models with predetermined variables. *Journal of Economic Dynamics and Control 29*(4), 601–625.

Farmer, R. E., V. Khramov, and G. Nicolò (2015). Solving and estimating indeterminate DSGE models. *Journal of Economic Dynamics and Control 54*, 17–36.

Fernández-Villaverde, J., J. F. Rubio-Ramírez, T. J. Sargent, and M. W. Watson (2007, June). ABCs (and Ds) of understanding VARs. *American Economic Review 97*(3), 1021–1026.

Hamilton, J. D. (1994). *Time-Series Analysis.* Princeton, NJ: Princeton University Press.

Hansen, L. P. and T. J. Sargent (2005, September). Recursive models of dynamic linear economies. unpublished Manuscript.

Hansen, L. P. and T. J. Sargent (2007). *Robustness.* Princeton University Press.

Kailath, T., A. H. Sayed, and B. Hassibi (2000). *Linear Estimation.* Prentice Hall Information and System Sciences Series. Pearson Publishing.

King, R. G. and M. W. Watson (1998, November). The solution of singular linear difference systems under rational expectations. *Internatinal Economic Review 39*(4), 1015–1026.

Klein, P. (2000, September). Using the generalized Schur form to solve a multivariate linear rational expectations model. *Journal of Economic Dynamics and Control 24*(10), 1405–1423.

Komunjer, I. and S. Ng (2011, November). Dynamic identification of dynamic stochastic general equilibrium models. *Econometrica 79*(6), 1995 – 2032.

Lubik, T. A. and C. Matthes (2016). Indeterminacy and learning: An analysis of monetary policy in the great inflation. *Journal of Monetary Economics 82*(C), 85–106.

Lubik, T. A. and F. Schorfheide (2003, November). Computing sunspot equilibria in linear rational expectations models. *Journal of Economic Dynamics and Control 28*(2), 273–285.

Lubik, T. A. and F. Schorfheide (2004, March). Testing for indeterminacy: An application to U.S. monetary policy. *The American Economic Review 94*(1), 190–217.

Mertens, E. (2016, June). Managing beliefs about monetary policy under discretion. *Journal of Money, Credit and Banking 48*(4), 661–698.

Orphanides, A. (2001, September). Monetary policy rules based on real-time data. *American Economic Review 91*(4), 964–985.

Orphanides, A. (2003). Monetary policy evaluation with noisy information. *Journal of Monetary Economics 50*(3), 605–631. Swiss National Bank/Study Center Gerzensee Conference on Monetary Policy under Incomplete Information.

Sargent, T. J. and L. Ljungqvist (2004). *Recursive Macroeconomic Theory* (2nd ed.). Cambridge, MA: The MIT Press.

Sims, C. A. (2002, October). Solving linear rational expectations models. *Computational Economics 20*(1-2), 1–20.

Svensson, L. E. O. and M. Woodford (2004, January). Indicator variables for optimal policy under asymmetric information. *Journal of Economic Dynamics and Control 28*(4), 661–690.