

Liquidity and Risk Management: Coordinating Investment and Compensation Policies^{*}

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Abstract

We formulate a theory of corporate liquidity and risk management for a firm run by a risk-averse entrepreneur, who cannot irrevocably commit her human capital. The firm's optimized balance sheet comprises on the asset side, illiquid capital, K , cash and marketable securities, S , and on the liability side, equity and an endogenous line of credit. The value-maximizing firm optimally smoothes compensation and manages its idiosyncratic and systematic risk exposures so as to retain managerial talent and maximize its investment efficiency. As the liquidity-to-capital ratio $s = S/K$ approaches the endogenous credit limit \underline{s} , the firm optimally responds by cutting investment and compensation, selling insurance, and reducing both the idiosyncratic and systematic volatilities of s . When the firm is less constrained, it invests more, increases compensation, engages more in financial hedging, and allocates liquid assets more in the market portfolio.

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1 Introduction

The Modigliani and Miller (MM) irrelevance theorem implies that firms cannot create any value through corporate liquidity and risk management. The basic logic is that any retained earnings or corporate hedging positions can be undone or replicated by the firm's investors, so that there is no value added in the firm doing this for them. Accordingly, first-generation theories of corporate liquidity and risk management have invoked financial market imperfections, such as tax distortions, or a wedge between internal and external funding costs, as a basic rationale for corporate risk management.¹

In this paper we develop a theory of liquidity and risk management that emphasizes benefits over and above those that spring from tax or asymmetric information distortions. These benefits have to do with the firm's enhanced ability through risk and liquidity management to retain talent and valuable human capital. Our theory considers the problem faced by a risk-averse entrepreneur, who cannot irrevocably commit her human capital to the firm. The entrepreneur has constant relative risk-averse preference and seeks to smooth consumption. To best retain the entrepreneur, the firm optimally compensates her with current and future promised consumption. But to back up these promises the firm must engage in liquidity and risk management.

The main point of departure of our analysis is that corporate risk management is not so much about achieving an optimal risk-return profile for investors, they can do that on their own, than to achieve optimal risk-return profiles for risk-averse, under-diversified, key employees (or the entrepreneur in our setting) under limited commitment. In effect, the firm is both the employer and the asset manager of its key employees. This perspective on corporate risk management is consistent with Duchin et al. (2016), who find that non-financial firms invest 40% of their liquid savings in risky financial assets. More strikingly, they find that the less constrained firms invest more in the market portfolio, which is consistent with our model's prediction that firms, if unconstrained, seek to attain the mean-variance frontier for their key employees. In contrast, we show that when severely constrained, firms cut compensation, reduce corporate investment, engage in asset sales, minimize exposures to the market and idiosyncratic risks, with the primary objective of surviving and retaining key employees. These results are also in line with the findings of Rampini, Sufi, and Viswanathan

¹See Stulz (1984), Smith and Stulz (1985), and Froot, Scharfstein, and Stein (1993).

(2014), Brown and Matsa (2016), and Donangelo (2016).

The firm's optimized balance sheet in our model is composed of illiquid capital, K , and cash and marketable securities, S , on the asset side. On the liability side, the firm has equity and a line of credit, with a limit that is endogenously determined. Illiquid capital can be accumulated subject to adjustment costs and is exposed to stochastic depreciation. The firm's operations are exposed to both idiosyncratic and aggregate risk. It manages its risk by choosing the optimal loadings on the idiosyncratic and market risk factors. The firm's liquidity is augmented via retained earnings from operations, profits and losses from its portfolio of marketable securities including its hedging and insurance positions. The unique state variable in our dynamic optimization problem is the firm's liquidity-to-capital ratio $s = S/K$. When liquidity s approaches an endogenously determined lower bound, where the firm's credit limit is exhausted, the entrepreneur optimally responds by cutting investment, consumption, and turning off all risk exposures of its liquidity s .

The model we develop generalizes the limited commitment framework of Hart and Moore (1994, 1998). They formulate a theory of debt and endogenous debt capacity arising from the inalienability of a risk-neutral entrepreneur's human capital. In a finite-horizon model with a single fixed project, deterministic cash flows, and fixed human capital, they show that there is a finite debt capacity for the firm, which is given by the maximum repayment that the entrepreneur can credibly promise: any higher repayment and the entrepreneur would abandon the firm. We generalize their model along the following dimensions: first, we introduce risky human capital and cash flows; second, we assume that the entrepreneur is risk averse; third, we consider an infinitely-lived firm with ongoing investment; and, fourth we also add a limited liability or commitment constraint for investors. In this more realistic model we are nevertheless able to derive the optimal investment, consumption, liquidity and risk management policy of the firm.

Rampini and Viswanathan (2010, 2013) also develop a limited-commitment-based theory of risk management, which focuses on the tradeoff between exploiting current versus future investment opportunities. If the firm invests today it may exhaust its debt capacity and thereby forego future investment opportunities. If instead the firm foregoes investment and hoards its cash it is in a position to be able to exploit potentially more profitable investment opportunities in the future. The main differences between our model and theirs are the risk aversion of the entrepreneur, the modeling of limited commitment in the form of risky

inalienable human capital, and the illiquidity of physical capital.

Moreover, we focus on a different aspect of corporate liquidity and risk management, namely the management of risky human capital. In particular, we emphasize the benefits of risk management to help smooth consumption of the firm's stakeholders (entrepreneur, managers, key employees). This consumption smoothing motive, which requires building liquidity buffers in low productivity states, is so strong that it generally outweighs the countervailing investment financing motive of Froot, Scharfstein, and Stein (1993) and Rampini and Viswanathan (2010, 2013), which requires building liquidity buffers in high productivity states, where investment opportunities are greatest. If the firm has been unable to build a sufficient liquidity buffer in the low productivity state, we show that it is optimal for the entrepreneur to take a pay cut, consistent with the evidence on executive compensation and corporate cash holdings (e.g. Ganor, 2013). It is also optimal to sell insurance on persistent productivity shocks to generate valuable liquidity. While asset sales in response to a negative productivity shock (also optimal in our setting) are commonly emphasized (Campello, Giambona, Graham and Harvey, 2011), our analysis further reveals the dynamic optimality of other observed corporate policies, such as selling insurance and moderating pay.

Our theory can in particular explain the observed corporate policies of human capital intensive, high-tech, firms. These firms often hold substantial cash pools, which may be necessary to make credible future compensation promises and thereby retain highly valued employees, who naturally have attractive alternative job opportunities. Indeed, employees in these firms are largely paid in the form of deferred stock compensation. When their stock options vest and are exercised, the companies often engage in stock repurchases so as to avoid excessive stock dilution. But such repurchase programs require funding, which could be part of the explanation for why these companies hold such large liquidity buffers.

The firm's optimal investment and consumption policies can be characterized as generalizations, that take account of the marginal corporate value of liquidity, of respectively, the classical q -theory of investment and the permanent-income theory of consumption.² Similarly, the firm's optimal liquidity and risk management policies can be characterized as a generalization of Merton's classical intertemporal portfolio-choice rules that again take into account the marginal value of liquidity.

²Faulkender and Wang (2006), Pinkowitz, Stulz, and Williamson (2006), Dittmar and Mahrt-Smith (2007), and Bolton, Schaller, and Wang (2014) find that the marginal value of cash is typically greater than one.

In sum, our model integrates the following four highly interdependent corporate policies: 1) liquidity management (cash and lines of credit); 2) systematic and idiosyncratic risk management; 3) executive compensation or consumption; 4) corporate investment and asset sales, which all must take account of the firm's endogenous credit limit \underline{s} , determined by the inalienability-of-human-capital constraints.

When the firm is flush with liquidity, it is effectively financially unconstrained, so that its investment approaches the Hayashi (1982) risk-adjusted first-best benchmark, and its consumption and asset allocations approach the (generalized) Merton (1971) consumption and mean-variance portfolio. In contrast, when the firm's liquidity is low, its primary concern is survival, which requires shutting down the volatility of the firm's liquidity s . Moreover, the firm substantially cuts back on all its expenditures and engages in asset and insurance sales to generate income.

We also show that the firm's optimal liquidity and risk management problem can be reformulated as a dual optimal contracting problem over consumption and investment between fully diversified investors and a risk-averse entrepreneur subject to inalienability-of-human-capital constraints. More concretely, in the dual optimal contracting problem the state variable is the certainty-equivalent wealth that investors promise to the entrepreneur, w , and the value of the firm to investors is $p(w)$. Moreover, under this optimal contract the firm's investment and financing policies and the entrepreneur's consumption are all expressed as functions of w . As Table 1 below summarizes, we show that this dual contracting problem is equivalent to the entrepreneur's liquidity and risk management problem with $s = -p(w)$ and the entrepreneur's certainty-equivalent value is $m(s) = w$. The key observation here is that the firm's endogenously determined credit limit is the outcome of an optimal financial contracting problem. In other words, the firm's financial constraint is an optimal credit limit that reflects the entrepreneur's inability to irrevocably commit her human capital to the firm.

We extend the simplest formulation of our model in two directions. First, we also introduce a limited commitment (or limited liability) constraint for investors. In the two-sided commitment problem, where a limited liability constraint for investors must also hold, we obtain further striking results. The firm may now over-invest and the entrepreneur may over-consume (compared to the first-best benchmark). The intuition is as follows. To make sure that investors do not default on their promises to the entrepreneur, w cannot exceed

Table 1: EQUIVALENCE: PRIMAL OPTIMIZATION AND DUAL CONTRACTING PROBLEMS

	Primal Optimization	Dual Contracting
State Variable	s	w
Value Function	$m(s)$	$p(w)$

an upper bound \bar{w} given by $p(\bar{w}) = 0$. In other words, the firm’s liquidity, s , cannot exceed $\bar{s} = 0$, otherwise, investors would simply syphon off the excess liquidity. As a result, the entrepreneur responds by increasing investment and consumption in order to satisfy the investors’ limited-commitment constraint.

Second, we introduce persistent productivity shocks in the form of a two-state Markov transition process between high and low productivity states. We show that the benefits of risk management to smooth consumption generally outweigh the countervailing investment financing benefits of allocating liquidity to high productivity states. In particular, it can be optimal for a highly financially constrained firm in the low productivity state to bet against the realization of a high productivity shock (in other words, to sell insurance) in order to generate liquidity.

Related literature. Our paper is related to the microeconomics literature on contracting under limited commitment following Harris and Holmstrom (1982). They analyze a model of optimal insurance for a risk-averse worker, who is unable to commit to a long-term contract. Berk, Stanton, and Zechner (2010) generalize Harris and Holmstrom (1982) by incorporating capital structure and human capital bankruptcy costs into their setting. In terms of methodology, our paper builds on the dynamic contracting in continuous time work of Holmstrom and Milgrom (1987), Schaettler and Sung (1993), and Sannikov (2008), among others.

Directly related to our theory is the contracting problem considered by Ai and Li (2015). They analyze a dynamic contracting problem between a risk-neutral shareholder and a risk-averse CEO that is similar to our dual contracting problem. They study the dynamics of optimal CEO compensation and investment under limited commitment and show that compensation increases as the firm becomes more profitable when the CEO’s participation constraint binds. They further show that the firm may over-invest when the limited com-

mitment constraint of the shareholder binds. Also closely related is Lambrecht and Myers (2012) who consider an intertemporal model of a firm run by a risk-averse entrepreneur with habit formation and derive the firm’s optimal dynamic corporate policies. They show that the firm’s optimal payout policy resembles the famous Lintner (1956) payout rule of thumb. Building on Merton’s intertemporal portfolio choice framework, Wang, Wang, and Yang (2012) study a risk-averse entrepreneur’s optimal consumption-savings, portfolio choice, and capital accumulation decisions when facing uninsurable capital and productivity risks. Unlike Wang, Wang, and Yang (2012), our model features optimal liquidity and risk management policies that arise endogenously from an underlying financial contracting problem.

Our paper is evidently related to the dynamic corporate security design literature in the vein of DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), and DeMarzo and Fishman (2007b).³ These papers also focus on the implementation of the optimal contracting solution via corporate liquidity (cash and credit line.) Two key differences between our model and these papers are (1) risk aversion and (2) systematic and idiosyncratic risk, which together lead to a theory of corporate portfolio management, thus explaining the “marketable securities” entry on corporate balance sheets plus the zero-NPV, off-balance-sheet hedging instruments (e.g. futures and insurance.) A third key difference is that these papers focus on moral hazard while we focus on the inalienability of risky human capital. A fourth difference is that we model corporate investment and generalize the q -theory of investment to settings with limited commitment.⁴

Our paper also relates to the macroeconomics literature that studies the implications of dynamic agency problems for firms’ investment and financing decisions. Green (1987), Thomas and Worrall (1990), Marcet and Marimon (1992), Kehoe and Levine (1993) and Kocherlakota (1996) are important early contributions on optimal contracting.

Building on these early contributions, Alvarez and Jermann (2000) extend the first and second welfare theorems to economies with limited commitment. Our entrepreneur’s optimization problem is closely related to the agent’s dynamic optimization problem in Alvarez and Jermann (2000). While their focus is on optimal consumption allocation, we focus

³See also Biais, Mariotti, Rochet, and Villeneuve (2010), and Piskorski and Tchisty (2010), among others. Biais, Mariotti, and Rochet (2013) and Sannikov (2013) provide recent surveys on this subject. For static security design models, see Townsend (1979) and Gale and Hellwig (1985), Innes (1990), and Holmstrom and Tirole (1997).

⁴DeMarzo and Fishman (2007a) and DeMarzo, Fishman, He and Wang (2012) generalize the moral hazard model of DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007b) to allow for investment.

on both consumption and corporate investment. As in DeMarzo and Sannikov (2006), the continuous-time formulation allows us to provide sharper, closed-form solutions for consumption, investment, liquidity and risk management policies up to an ordinary differential equation (ODE) for the entrepreneur’s certainty equivalent wealth $m(s)$.

Albuquerque and Hopenhayn (2004), Quadrini (2004), and Clementi and Hopenhayn (2006) characterize firms’ financing and investment decisions under limited commitment and/or asymmetric information.⁵ Kehoe and Perri (2002) and Albuquerque (2003) analyze the implications of limited commitment in international business cycles and foreign direct investments. Lorenzoni and Walentin (2007) study q theory of investment under limited commitment. Miao and Zhang (2015) develop a duality-based solution method for limited commitment problems. Finally, our paper is clearly related to the voluminous economics literature on human capital that builds on Ben-Porath (1967) and Becker (1975).

2 Model

We consider an intertemporal optimization problem faced by a risk-averse entrepreneur, who optimally chooses her consumption, savings, capital investment, and exposures to both systematic and idiosyncratic risks, subject to the limited commitment constraint that she cannot promise to operate the firm indefinitely under any circumstances. This limited-commitment problem for the entrepreneur results in an *endogenous* financial constraint for the firm. To best highlight the central economic mechanism arising from this limited-commitment constraint, we remove all other financial frictions from the model and assume that financial markets are otherwise fully competitive and dynamically complete (we show how dynamic completeness is constructed through spanning in Section 2.2). The detailed model description begins below with the entrepreneur’s production technology and preferences.

2.1 Production Technology and Preferences

Production and Capital Accumulation. We adopt the capital accumulation specification of Cox, Ingersoll, and Ross (1985) and Jones and Manuelli (2005). The firm’s capital

⁵See Ljungqvist and Sargent (2004) Part V for a textbook treatment on the limited-commitment-based macro models.

stock K evolves according to a controlled Geometric Brownian Motion (GBM) process:

$$dK_t = (I_t - \delta_K K_t)dt + \sigma_K K_t \left(\sqrt{1 - \rho^2} d\mathcal{Z}_{1,t} + \rho d\mathcal{Z}_{2,t} \right), \quad (1)$$

where I is the firm's rate of gross investment, $\delta_K \geq 0$ is the expected rate of depreciation, and σ_K is the volatility of the capital depreciation shock. Without loss of generality, we decompose risk into two orthogonal components: an idiosyncratic shock represented by the standard Brownian motion \mathcal{Z}_1 and a systematic shock represented by the standard Brownian motion \mathcal{Z}_2 . The parameter ρ measures the correlation between the firm's capital risk and systematic risk, so that the firm's systematic volatility is equal to $\rho\sigma_K$ and its idiosyncratic volatility is given by

$$\epsilon_K = \sigma_K \sqrt{1 - \rho^2}. \quad (2)$$

Production requires combining the entrepreneur's inalienable human capital with the firm's capital stock K_t , which together yield revenue AK_t . Without the entrepreneur's human capital the capital stock K_t does not generate any cash flows.⁶ Investment involves both a direct purchase and an adjustment cost, so that the firm's *free cash flow* (after capital expenditures) is given by:

$$Y_t = AK_t - I_t - G(I_t, K_t), \quad (3)$$

where the price of the investment good is normalized to one and $G(I, K)$ is the standard adjustment cost function in the q -theory of investment. Note that Y_t can take negative values, which simply means that investors provide additional financing to close the gap between contemporaneous revenue, AK_t , and investment and compensation outlays. We simplify the model by assuming that the firm's adjustment cost $G(I, K)$ is homogeneous of degree one in I and K (a common assumption in the q -theory of investment), so that $G(I, K)$ takes the following separable form:

$$G(I, K) = g(i)K, \quad (4)$$

where $i = I/K$ denotes the firm's investment-capital ratio and $g(i)$ is increasing and convex in i . As Hayashi (1982) has shown, given this homogeneity property Tobin's average and

⁶An implication of our assumptions is that managerial retention is always optimal as we show in our optimal contracting formulation.

marginal q are equal in the First-best benchmark.⁷ However, under limited commitment an endogenous wedge between Tobin’s average and marginal q will emerge in our model.⁸

Preferences. The infinitely-lived entrepreneur has a standard concave utility function over positive consumption flows $\{C_t; t \geq 0\}$ given by:

$$J_t = \mathbb{E}_t \left[\int_t^\infty \zeta e^{-\zeta(v-t)} U(C_v) dv \right], \quad (5)$$

where $\zeta > 0$ is the entrepreneur’s subjective discount rate, $\mathbb{E}_t[\cdot]$ is the time- t conditional expectation, and $U(C)$ takes the standard constant-relative-risk-averse utility (CRRA) form:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}, \quad (6)$$

with $\gamma > 0$ denoting the coefficient of relative risk aversion. We normalize the flow payoff with ζ in (5), so that the utility flow is given by $\zeta U(C)$.⁹

2.2 Complete Financial Markets

We assume that financial markets are perfectly competitive and complete. Market completeness is obtained through dynamic spanning with three long-lived assets as in the Black-Merton-Scholes framework (Duffie and Huang, 1985): Given that the firm’s production is subject to two shocks, \mathcal{Z}_1 and \mathcal{Z}_2 , financial markets are dynamically complete if the following three non-redundant financial assets can be dynamically and frictionlessly traded:

- a. A risk-free asset that pays interest at a constant risk-free rate r ;

⁷Lucas and Prescott (1971) analyze dynamic investment decisions with convex adjustment costs, though they do not explicitly link their results to marginal or average q . Abel and Eberly (1994) extend Hayashi (1982) to a stochastic environment and a more general specification of adjustment costs.

⁸An endogenous wedge between Tobin’s average and marginal q also arises in cash-based optimal financing and investment models such as Bolton, Chen, and Wang (2011) and optimal contracting models such as DeMarzo, Fishman, He, and Wang (2012).

⁹This normalization is convenient in contracting models (see Sannikov (2008)). We can generalize these preferences to allow for a coefficient of relative risk aversion that is different from the inverse of the elasticity of intertemporal substitution, à la Epstein and Zin (1989). Indeed, as Epstein-Zin preferences are homothetic, allowing for such preferences in our model will not increase the dimensionality of the optimization problem. Details are available upon request.

- b. A risky asset that is perfectly correlated with the idiosyncratic shock \mathcal{Z}_1 . The incremental return $dR_{1,t}$ on this risky asset over the time interval dt is

$$dR_{1,t} = rdt + \epsilon_K d\mathcal{Z}_{1,t}. \quad (7)$$

Note that the expected return on this risky asset equals the risk-free rate r . As it is only subject to idiosyncratic shocks it earns no risk premium. We let the volatility of this risky asset be ϵ_K without loss of generality;

- c. A risky asset that is perfectly correlated with the systematic shock \mathcal{Z}_2 . The incremental return $dR_{2,t}$ of this asset over the time interval dt is

$$dR_{2,t} = \mu_2 dt + \sigma_2 d\mathcal{Z}_{2,t}, \quad (8)$$

where μ_2 and σ_2 are constant mean and volatility parameters. As this risky asset is only subject to the systematic shock we refer to it as the *market portfolio*.

Dynamic and frictionless trading with these three securities implies that the following unique stochastic discount factor (SDF) exists:

$$\frac{d\mathbb{M}_t}{\mathbb{M}_t} = -rdt - \eta d\mathcal{Z}_{2,t}, \quad (9)$$

where $\mathbb{M}_0 = 1$ and η is the Sharpe ratio of the market portfolio given by:

$$\eta = \frac{\mu_2 - r}{\sigma_2}.$$

Note that the SDF \mathbb{M} follows a geometric Brownian motion with the drift equal to the negative risk-free rate, as required under no-arbitrage. By definition the SDF is only exposed to the systematic shock \mathcal{Z}_2 . Fully diversified investors will only demand a risk premium for their exposures to systematic shocks. The entrepreneur, however, is not fully diversified given her exposure to the risky venture.

2.3 Limited Commitment and Endogenous Borrowing Capacity

The entrepreneur has an option to walk away at any time from her current firm of size K_t , thereby leaving behind all her liabilities. What deters the entrepreneur from doing so is the fact that she is more efficient at her current firm than at her next-best alternative, a firm of size αK_t , where $\alpha \in (0, 1)$ is a given constant. Thus, as long as the current firm's liabilities are not too large the entrepreneur prefers to stay with the firm. Note that we are expressing the limited commitment constraint in the form of an alternative firm where the entrepreneur's human capital can be deployed. Under this formulation of the entrepreneur's outside option there is no need for misappropriation of the firm's capital stock by the entrepreneur. An alternative interpretation of the entrepreneur's outside option is, of course, that she can at any time t abscond with a fraction α of the firm's capital stock K_t and start afresh with zero liabilities.¹⁰

The limited-commitment constraints naturally map into an endogenous debt capacity for the firm as in Hart and Moore (1994, 1998), Kiyotaki and Moore (1997), and Albuquerque and Hopenhayn (2004). However, unlike these models our framework incorporates both idiosyncratic and aggregate shocks. Additionally, we link the determinants of the firm's endogenous debt capacity to not only corporate investment and asset sales, but also corporate liquidity, risk management with respect to idiosyncratic and aggregate shocks, and optimal compensation. Before characterizing the solution under limited commitment, we derive the first-best optimum under full commitment.

3 First Best

Under dynamically complete markets the entrepreneur's savings, portfolio allocation, and consumption problem, to maximize her utility, can be separated from the corporate investment problem, to maximize firm value (see Duffie, 2001). There are two ways of formulating the first-best optimization model: either as a static maximization problem with a single intertemporal budget constraint, or as a dynamic programming problem with continuous, dynamic, portfolio rebalancing. The latter construction provides a more direct link to the

¹⁰In practice entrepreneurs can sometimes partially commit themselves and lower their outside options by signing non-compete clauses. This possibility can be captured in our model by lowering the parameter α , which relaxes the entrepreneur's inalienability-of-human-capital constraints.

problem under limited commitment, since it is the limit formulation when the entrepreneur's commitment constraint vanishes. Accordingly, we shall rely on the dynamic programming method to characterize the first-best solution, which can be framed without loss of generality as a dynamic liquidity and risk management problem.

3.1 First-Best Liquidity and Risk Management Problem

The entrepreneur's total wealth includes both her *liquid* financial holdings and her ownership of the *illiquid* productive capital K . Let $\{S_t : t \geq 0\}$ denote the entrepreneur's liquid wealth process. The entrepreneur continuously allocates $\{S_t : t \geq 0\}$ to any admissible positions $\{\Phi_{1,t}, \Phi_{2,t} : t \geq 0\}$ in the two risky financial assets, whose returns are given by (7) and (8) respectively, and the residual amount $(S_t - \Phi_{1,t} - \Phi_{2,t})$ to the risk-free asset. Her liquid wealth then stochastically evolves as follows:

$$dS_t = (rS_t + Y_t - C_t)dt + \Phi_{1,t}\epsilon_K d\mathcal{Z}_{1,t} + \Phi_{2,t}[(\mu_2 - r)dt + \sigma_2 d\mathcal{Z}_{2,t}]. \quad (10)$$

The first term in (10), $rS_t + Y_t - C_t$, is simply the sum of the firm's interest income rS_t and net operating cash flows, $Y_t - C_t$, the second term, $\Phi_{1,t}\epsilon_K d\mathcal{Z}_{1,t}$, is the exposure to the idiosyncratic shock \mathcal{Z}_1 , which earns no risk premium, and the third term, $\Phi_{2,t}[(\mu_2 - r)dt + \sigma_2 d\mathcal{Z}_{2,t}]$, is the excess return from the investment in the market portfolio.

In the absence of any risk exposure $rS_t + Y_t - C_t$ is simply the rate at which the entrepreneur saves when $S_t \geq 0$ or dissaves (by drawing on a line of credit (LOC) at the risk-free rate r , when $S_t < 0$). In general, saving all liquid wealth S at the risk-free rate is sub-optimal. By dynamically engaging in risk taking and risk management, through the risk exposures Φ_1 and Φ_2 , the entrepreneur will do better, as we show next.

The Entrepreneur's Optimization Problem. The entrepreneur dynamically chooses consumption C , corporate investment I , idiosyncratic risk hedging demand Φ_1 , and the market portfolio exposure Φ_2 , to maximize her utility given in (5)-(6), subject to the liquidity accumulation dynamics (10) and the transversality condition $\lim_{v \rightarrow \infty} \mathbb{E}_t [e^{-\zeta(v-t)} |J_v|] = 0$, where J_v is the entrepreneur's time- v value function.

Let $J(K_t, S_t)$ denote the entrepreneur's time- t value function J_t which depends on the firm's capital stock K_t and liquid savings S_t . By the standard dynamic programming argu-

ment, $J(K, S)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} \zeta J(K, S) = & \max_{C, I, \Phi_1, \Phi_2} \zeta U(C) + (rS + \Phi_2(\mu_2 - r) + AK - I - G(I, K) - C)J_S(K, S) \\ & + (I - \delta_K K)J_K + \frac{\sigma_K^2 K^2}{2} J_{KK}(K, S) + (\epsilon_K^2 \Phi_1 + \rho \sigma_K \sigma_2 \Phi_2) K J_{KS}(K, S) \\ & + \frac{(\epsilon_K \Phi_1)^2 + (\sigma_2 \Phi_2)^2}{2} J_{SS}(K, S). \end{aligned} \quad (11)$$

The first term on the right side of (11) represents the entrepreneur's normalized flow utility of consumption; the second term (involving J_S) represents the marginal value of incremental liquid savings S ; the third term (involving J_K) represents the marginal value of net investment $(I - \delta_K K)$; and the last three terms (involving J_{KK} , J_{KS} , and J_{SS}) capture the impact of idiosyncratic and systematic shocks.

Given the concavity of the utility function $U(C)$ and the convexity of the capital adjustment cost function, the optimal consumption $C(K, S)$ and investment $I(K, S)$ rules are characterized by the following first-order conditions (FOCs):

$$\zeta U'(C) = J_S(K, S), \quad (12)$$

and

$$1 + G_I(I, K) = \frac{J_K(K, S)}{J_S(K, S)}. \quad (13)$$

Equation (12) is the standard FOC for consumption, equating the marginal utility of consumption with the marginal value of savings J_S , and equation (13) states that the marginal cost of investing $(1 + G_I)$ is equal to the entrepreneur's marginal value of investing, which is given by the ratio J_K/J_S of the entrepreneur's marginal value of illiquid capital J_K to the marginal value of liquid savings J_S .

Similarly, we obtain the following FOCs for the firm's optimal risk exposure policies:

$$\Phi_1 = -\frac{K J_{KS}}{J_{SS}}, \quad (14)$$

and

$$\Phi_2 = -\frac{\eta}{\sigma_2} \frac{J_S}{J_{SS}} - \frac{\rho \sigma_K}{\sigma_2} \frac{K J_{KS}}{J_{SS}}. \quad (15)$$

We refer to Φ_1 as the *idiosyncratic-risk hedging demand*: the only reason for holding this risky asset is for hedging purposes against the firm's idiosyncratic risks. The entrepreneur's market portfolio holding Φ_2 is given by the classical exposure to the market excess return (the first term) and a hedge against the firm's systematic-risk exposure (the second term). Equations (11), (12), (13), (14) and (15) jointly characterize the solution to the entrepreneur's optimization problem.

Guided by the observation that the value function for the standard Merton portfolio-choice problem (without illiquid assets) inherits the CRRA form of the agent's utility function $U(\cdot)$, we conjecture and verify that the entrepreneur's value function in the first-best problem, denoted by $J^{FB}(K, S)$, takes the same form as in Merton's problem:

$$J^{FB}(K, S) = \frac{(bM^{FB}(K, S))^{1-\gamma}}{1-\gamma}, \quad (16)$$

where $M^{FB}(K, S)$ is the market value of the entrepreneur's wealth (to be derived) and b is the following constant:¹¹

$$b = \zeta \left[\frac{1}{\gamma} - \frac{1}{\zeta} \left(\frac{1-\gamma}{\gamma} \right) \left(r + \frac{\eta^2}{2\gamma} \right) \right]^{\frac{\gamma}{\gamma-1}}. \quad (17)$$

To ensure that the problem is well posed under first best, we require the following parameter constraint:

$$\text{Condition 1: } \frac{1}{\gamma} - \frac{1}{\zeta} \left(\frac{1-\gamma}{\gamma} \right) \left(r + \frac{\eta^2}{2\gamma} \right) > 0. \quad (18)$$

3.2 Solution

Given that markets are complete, the firm's investment decisions are chosen to maximize the market value of its capital stock. Since our model is homogeneous in K , we can characterize the solution per unit of capital. As is standard, we therefore describe the solution in terms of lower-case variables: consumption $c_t = C_t/K_t$, investment $i_t = I_t/K_t$, liquidity, $s_t = S_t/K_t$, capital adjustment cost $g_t = G_t/K_t$, idiosyncratic risk hedging demand $\phi_1 = \Phi_1/K$, and market portfolio demand $\phi_2 = \Phi_2/K$.

¹¹In the special case when $\gamma = 1$ we have $b = \zeta \exp \left[\frac{1}{\zeta} \left(r + \frac{\eta^2}{2} - \zeta \right) \right]$.

First, we show that the value of capital Q_t^{FB} follows a GBM process given by:

$$dQ_t^{FB} = Q_t^{FB} [(i^{FB} - \delta_K) dt + (\epsilon_K d\mathcal{Z}_{1,t} + \rho\sigma_K d\mathcal{Z}_{2,t})] ,$$

with the drift $(i^{FB} - \delta_K)$, idiosyncratic volatility ϵ_K , and systematic volatility $\rho\sigma_K$, identical to those for the dynamics for $\{K_t : t \geq 0\}$.

Corporate Investment, the Value of Capital Q^{FB} , and Asset Pricing. The following proposition characterizes the first-best implications for corporate investment, the value of capital, and asset pricing.

Proposition 1 *The value of capital, $Q^{FB}(K)$, is proportional to K , $Q^{FB}(K) = q^{FB}K$, where q^{FB} is Tobin's q solving:*

$$q^{FB} = \max_i \frac{A - i - g(i)}{r + \delta - i} , \quad (19)$$

and the maximand for (19), denoted by i^{FB} , is the first-best investment-capital ratio. The risk-adjusted capital depreciation rate, δ equals the expected depreciation rate δ_K augmented by the risk premium $\rho\eta\sigma_K$:

$$\delta = \delta_K + \rho\eta\sigma_K . \quad (20)$$

The derivations for this and all other propositions in the remainder of the paper are provided in the Appendix.

This proposition generalizes the well known Hayashi conditions linking investment to Tobin's average (and marginal) q , by extending his framework to situations where the firm's operations are subject to both idiosyncratic and systematic risk, and where systematic risk commands a risk premium. As in the q -theory of investment, capital adjustment costs create a wedge between the value of installed capital and newly purchased capital, so that $q^{FB} \neq 1$ in general. Optimal investment i is given by the solution to the FOC for investment:

$$q^{FB} = 1 + g'(i^{FB}) , \quad (21)$$

which equates marginal q to the marginal cost of investing $1 + g'(i)$, at the optimum investment level i^{FB} . Jointly solving (19) and (21) yields the values for q^{FB} and i^{FB} . To ensure

that the value of capital Q_t^{FB} is finite, which is necessary for convergence under first best, we assume that the following condition is satisfied:

$$\text{Condition 2: } i^{FB} < r + \delta. \quad (22)$$

where i^{FB} is the unique maximand for (19). Let μ^{FB} denote the expected market return for the value of capital, Q_t^{FB} . Using Ito's formula, we may then express the expected return μ^{FB} as:

$$\mu^{FB} = \frac{A - i^{FB} - g(i^{FB})}{q^{FB}} + (i^{FB} - \delta_K) = r + \delta - i^{FB} + (i^{FB} - \delta_K) = r + \beta^{FB} (\mu_2 - r), \quad (23)$$

where the first equality gives the sum of the dividend yield and expected capital gains, the second equality uses (19), and where

$$\beta^{FB} = \frac{\rho\sigma_K}{\sigma_2}. \quad (24)$$

That is, the CAPM holds for the value of capital Q_t^{FB} , where β^{FB} is given by (24).

Optimal Consumption $c^{FB}(s)$ and Asset Allocation $(\phi_1^{FB}(s), \phi_2^{FB}(s))$. The next proposition characterizes optimal consumption and asset allocation rules.

Proposition 2 *Under Condition 1 and Condition 2 given by (18) and (22), the entrepreneur's optimal consumption policy is given by*

$$c^{FB}(s) = \chi (s + q^{FB}), \quad (25)$$

where χ is the marginal propensity to consume (MPC) given by

$$\chi = r + \frac{\eta^2}{2\gamma} + \gamma^{-1} \left(\zeta - r - \frac{\eta^2}{2\gamma} \right). \quad (26)$$

The first-best idiosyncratic risk hedge ϕ_1^{FB} and the market portfolio allocation ϕ_2^{FB} are re-

spectively given by:

$$\phi_1^{FB}(s) = -q^{FB}, \quad (27)$$

$$\phi_2^{FB}(s) = -\beta^{FB}q^{FB} + \frac{\eta}{\gamma\sigma_2}(s + q^{FB}). \quad (28)$$

Under the first best, the entrepreneur's total net worth, denoted by M_t^{FB} , is given by the sum of her liquid wealth S_t and the market value of capital $Q^{FB}(K_t)$:

$$M_t^{FB} = Q^{FB}(K_t) + S_t = q^{FB}K_t + S_t. \quad (29)$$

Again using Ito's formula, we can express the dynamics of $\{M_t^{FB} : t \geq 0\}$ as:

$$dM_t^{FB} = M_t^{FB} \left[\left(r - \chi + \frac{\eta^2}{\gamma} \right) dt + \frac{\eta}{\gamma} d\mathcal{Z}_{2,t} \right]. \quad (30)$$

That is, total net worth M is a GBM process with drift $(r - \chi) + \eta^2/\gamma$ and volatility η/γ for the systematic shock \mathcal{Z}_2 .

Two important observations follow from this characterization: First, we note that total net worth M^{FB} has zero net exposure to the idiosyncratic risk \mathcal{Z}_1 . This is simply due to the fact that the entrepreneur is averse to any net exposure to risk that does not generate any risk premium. How does the entrepreneur achieve this? One way is for her to take an offsetting short idiosyncratic risk exposure in the financial markets by setting $\phi_1^{FB} = -q^{FB}$, so that her exposure to the idiosyncratic risk \mathcal{Z}_1 through her long position in the business venture is exactly offset by an equivalent short position in the financial asset that is exposed to the idiosyncratic risk \mathcal{Z}_1 .

Second, under perfect and complete financial markets, the entrepreneur capitalizes the entire present value of her capital stock K at a unit price of q^{FB} . She then constructs a Merton-type consumption and portfolio allocation that results in net worth M^{FB} . That is why the marginal propensity to consume (MPC) and the dynamics for the net worth M^{FB} are the same as those in Merton (1971).

In summary, our first-best benchmark has the following important characteristics: 1) An optimal consumption rule that is linear in total net worth M^{FB} ; 2) An optimal liquidity and risk management policy such that the entrepreneur's net exposure to idiosyncratic risk

is entirely eliminated as seen from (30), and net exposure to systematic risk of η/γ , as in Merton (1971); 3) A constant investment-capital ratio and a constant Tobin's q as in Hayashi (1982); 4) An endogenous value for the capital process Q^{FB} that follows a GBM process as in the Black-Scholes economy.

4 Solution under Limited Commitment

Next, we characterize the entrepreneur's optimization problem as a liquidity and risk management problem under limited commitment. The entrepreneur's inability to fully commit will constrain her ability to dynamically manage liquidity and risk over time and across states of nature, in particular by limiting her credit capacity. The entrepreneur responds to the constraints on her ability to obtain an optimal risk exposure through financial markets, by engaging in self-insurance through liquidity management.

Limited Commitment and Endogenous Credit Capacity. The entrepreneur can at any moment walk away from her firm of size K_t , leaving behind all her liabilities. What deters her from doing so is that she is more efficient and better off with her current firm than her next-best alternative, a firm with size αK_t (where $\alpha \in (0, 1)$ is a given constant), provided that the firm's current liabilities are not too high. More formally, the firm's endogenous debt capacity, denoted by \underline{S}_t , satisfies the following equation:

$$J(K_t, \underline{S}_t) = J(\alpha K_t, 0), \quad (31)$$

which equates the value for the entrepreneur $J(K, \underline{S})$ of remaining with the firm and her outside option $J(\alpha K, 0)$. Given that it is never efficient for the entrepreneur to quit on the equilibrium path, $J(K, S)$ must satisfy the following condition:

$$J(K_t, S_t) \geq J(K_t, \underline{S}_t). \quad (32)$$

Using arguments similar to Alvarez and Jermann (2000) and Ai and Li (2015), we can show that the entrepreneur's optimization problem is increasing and concave.¹² Therefore, we can

¹²Due to space constraints, we omit the proofs. They are available upon request from the authors.

write (31) and (32) as:

$$S_t \geq \underline{S}_t = \underline{S}(K_t), \quad (33)$$

where $\underline{S}(K_t)$ defines the firm's endogenous credit capacity for any given capital stock K_t . When $S_t < 0$, the entrepreneur is in debt and draws down a LOC. The entrepreneur can borrow on this LOC at the risk-free rate r up to $\underline{S}(K_t)$ because (33) ensures that the entrepreneur does not walk away from the firm in an attempt to evade her debt obligations. Finally, we may write the endogenous debt capacity constraint (33) as follows by appealing to our model's homogeneity property in K :

$$s_t \geq \underline{s}, \quad (34)$$

where $|\underline{s}|$ is the endogenous credit capacity per unit of capital.

The Entrepreneur's Optimization Problem and Certainty-Equivalent Wealth.

Other than facing the additional endogenous credit constraint discussed above, the entrepreneur faces essentially the same tradeoffs in the interior region as in the first-best problem of Section 3.1. In particular, the FOCs for C , I , Φ_1 , and Φ_2 in the limited commitment problem are given by (12), (13), (14), and (15), respectively.

Again guided by the observation that the entrepreneur's value function inherits the CRRA form of her utility function $U(\cdot)$ in the first-best problem, we conjecture and verify that the entrepreneur's value function under limited commitment $J(K, S)$ also takes the form:

$$J(K, S) = \frac{(bM(K, S))^{1-\gamma}}{1-\gamma}, \quad (35)$$

where now $M(K, S)$ is the certainty-equivalent wealth of the entrepreneur in the limited commitment problem. Note that b in (35) is the same as in (17), the value of b in the first-best problem. Because our model is homogeneous in K and S , we can express the certainty equivalent wealth function per unit of capital:

$$M(K, S) = m(s) \cdot K. \quad (36)$$

Optimal Consumption $c(s)$, Investment $i(s)$, and Asset Allocation $\phi_1(s)$ and $\phi_2(s)$. The next proposition characterizes the optimal consumption, investment, and asset allocation rules under limited commitment in the interior region where $s > \underline{s}$.

Proposition 3 *Given $m(s)$, the optimal consumption policy is*

$$c(s) = \chi m'(s)^{-\gamma} m(s), \quad (37)$$

where χ is the MPC under first best given by (26). Corporate investment satisfies:

$$1 + g'(i) = \frac{m(s)}{m'(s)} - s. \quad (38)$$

The idiosyncratic risk hedge $\phi_1(s)$ and the market portfolio allocation $\phi_2(s)$ are given by:

$$\phi_1(s) = -\frac{J_{KS}}{J_{SS}} = \frac{sm''(s)m(s) + \gamma m'(s)(m(s) - sm'(s))}{m(s)m''(s) - \gamma m'(s)^2}, \quad (39)$$

$$\begin{aligned} \phi_2(s) &= -\frac{\eta}{\sigma_2} \frac{J_S}{K J_{SS}} - \beta^{FB} \frac{J_{KS}}{J_{SS}}, \\ &= \beta^{FB} \frac{sm''(s)m(s) + \gamma m'(s)(m(s) - sm'(s))}{m(s)m''(s) - \gamma m'(s)^2} - \frac{\eta}{\sigma_2} \frac{m'(s)m(s)}{m(s)m''(s) - \gamma m'(s)^2}. \end{aligned} \quad (40)$$

Endogenous Credit Limit and the Dynamics of Liquidity s . Given policy rules $c(s)$, $i(s)$, $\phi_1(s)$, and $\phi_2(s)$, and using Ito's formula, we show that in the interior region, where $s_t > \underline{s}$, liquidity s evolves according to the accumulation equation:

$$ds_t = \mu^s(s_t)dt + \sigma_1^s(s_t)d\mathcal{Z}_{1,t} + \sigma_2^s(s_t)d\mathcal{Z}_{2,t}, \quad (41)$$

where the drift function $\mu^s(\cdot)$ is given by:

$$\begin{aligned} \mu^s(s) &= A - i(s) - g(i(s)) + \phi_2(s)(\mu_2 - r) - c(s) \\ &\quad + (r + \delta_K - i(s))s - (\epsilon_K \sigma_1^s(s) + \rho \sigma_K \sigma_2^s(s)), \end{aligned} \quad (42)$$

and the idiosyncratic volatility $\sigma_1^s(\cdot)$ and systematic volatility $\sigma_2^s(\cdot)$ for s are given by

$$\sigma_1^s(s) = (\phi_1(s) - s)\epsilon_K, \quad (43)$$

$$\sigma_2^s(s) = (\phi_2(s) - s\beta^{FB})\sigma_2. \quad (44)$$

Having described the dynamics for s in the interior region $s_t > \underline{s}$, we next turn to the *endogenous* credit limit \underline{s} . Substituting the value function (35) into (31) and using homogeneity, we obtain the following condition for the credit limit \underline{s} :

$$m(\underline{s}) = \alpha m(0). \quad (45)$$

An important first observation is that the constraint $s_t \geq \underline{s}$ generally does not bind. The reason is that, as in the buffer-stock savings models of Deaton (1991) and Carroll (1997) for household finance, the *risk-averse* entrepreneur manages her liquid holdings s with the objective of smoothing her consumption. Setting $s_t = \underline{s}$ for all t is costly in terms of consumption smoothing. This is why risk aversion plays an important role in our model of liquidity and risk management.

Second, while the credit constraint $s_t \geq \underline{s}$ rarely binds, it has to be satisfied with probability one. Only then can we ensure that the credit limit is never exceeded and that the entrepreneur does not default. Given that s is a diffusion process and hence is continuous, to ensure that the firm's debt amount does not exceed its credit limit $|\underline{s}|$, intuitively speaking, we must require that the volatility at \underline{s} is zero, so that

$$\sigma_1^s(\underline{s}) = 0 \quad \text{and} \quad \sigma_2^s(\underline{s}) = 0. \quad (46)$$

The above boundary conditions are expressed in terms of volatility functions $\sigma_1^s(\cdot)$ and $\sigma_2^s(\cdot)$, which is somewhat unconventional but it helps to bring out the intuition. More conventionally, we can express these boundary conditions at \underline{s} equivalently in terms of the value function and its derivatives. For our problem, (46) boils down to

$$\lim_{s \rightarrow \underline{s}} m''(s) = -\infty. \quad (47)$$

We can also show this result via the regulated Brownian motion argument as Proposition 7 on page 84 in Harrison (1985).¹³ Moreover, we need to verify that the drift $\mu^s(s)$ given in (42) is non-negative at \underline{s} , so that liquidity s is weakly increasing at \underline{s} with probability one.

Finally, as $s \rightarrow \infty$, the firm is no longer credit constrained so that the limited-commitment certainty equivalent value approaches the first-best market value:

$$\lim_{s \rightarrow \infty} m(s) = q^{FB} + s. \quad (48)$$

The following proposition summarizes the limited-commitment solution for $m(s)$.

Proposition 4 *In the interior region $s > \underline{s}$, $m(s)$ satisfies the following ODE:*

$$0 = \frac{m(s)}{1-\gamma} \left[\gamma \chi m'(s)^{\frac{\gamma-1}{\gamma}} - \zeta \right] + [rs + A - i(s) - g(i(s))] m'(s) + (i(s) - \delta)(m(s) - sm'(s)) - \left(\frac{\gamma \sigma_K^2}{2} - \rho \eta \sigma_K \right) \frac{m(s)^2 m''(s)}{m(s) m''(s) - \gamma m'(s)^2} + \frac{\eta^2 m'(s)^2 m(s)}{2(\gamma m'(s)^2 - m(s) m''(s))}, \quad (49)$$

subject to the FOCs (37), (38), (39), and (40), the boundary condition (47) at \underline{s} , where \underline{s} satisfies (45), and condition (48).

Before continuing with a quantitative illustration of the entrepreneur's problem under limited commitment it is helpful to underline how the entrepreneur's dynamic liquidity and risk-management problem can be formulated equivalently as an optimal contracting problem between a diversified investor and a risk-averse entrepreneur subject to inalienability-of-human-capital constraints.

5 Equivalent Optimal Contract

Consider the long-term contracting problem between infinitely-lived fully diversified investors (*the principal*) and a financially constrained, infinitely-lived, risk-averse entrepreneur (*the agent*). Suppose that the output process Y_t is publicly observable and verifiable. In addition, suppose that the entrepreneur cannot privately save.¹⁴ The contracting game begins at time

¹³See Ai and Li (2015) for similar boundary conditions and proofs for their optimal contracting problem.

¹⁴This is a standard assumption in the dynamic moral hazard literature (Ch. 10 in Bolton and Dewatripont, 2005). Di Tella and Sannikov (2016) develop a contracting model with hidden savings for asset management.

0 with investors making a take-it-or-leave-it long-term contract offer to the entrepreneur. The contract specifies an investment process $\{I_t; t \geq 0\}$ and a compensation/consumption process $\{C_t; t \geq 0\}$ to the entrepreneur, both of which depend on the entire history of idiosyncratic and aggregate shocks $\{\mathcal{Z}_{1,t}, \mathcal{Z}_{2,t}; t \geq 0\}$.

Because investors are full diversified and markets are complete, their time-0 problem is to choose investment $\{I_t; t \geq 0\}$ and consumption $\{C_t; t \geq 0\}$ for the entrepreneur to maximize the risk-adjusted discounted value of future cash flows:

$$F(K_0, V_0) = \max_{C, I} \mathbb{E}_0 \left[\int_0^\infty \mathbb{M}_t (Y_t - C_t) dt \right], \quad (50)$$

where \mathbb{M} is the same unique SDF given in (9) and $\mathbb{M}_0 = 1$. The optimization is subject to the entrepreneur's time-0 participation constraint and her inalienability-of-human-capital constraints at all t , to which we now turn.

Inalienability-of-Human-Capital Constraints. The entrepreneur's human capital is inalienable and she can at any time leave the firm. Let $\widehat{V}(K_t)$ denote her (endogenous) outside payoff. Then the inalienability-of-human-capital constraint at time t is given by:

$$V_t \geq \widehat{V}(K_t), \quad t \geq 0. \quad (51)$$

Note that under this formulation the entrepreneur's inside value, V_t , and outside option value, $\widehat{V}(K_t)$, are both tied to the state variable K . This is why the entrepreneur's human capital is risky. Next, we determine $\widehat{V}(K_t)$ as follows: Consider the entrepreneur's next best-alternative which is to manage a new firm with size αK_t and no liabilities. Let $\overline{V}(\cdot)$ be the manager's value function in this new firm. Assume that investors in the new firm make zero profits under competitive markets, so that the following break-even condition holds:

$$F(\alpha K_t, \overline{V}(\alpha K_t)) = 0. \quad (52)$$

The entrepreneur's outside option value then satisfies:

$$\widehat{V}(K_t) = \overline{V}(\alpha K_t). \quad (53)$$

In words, equation (53) means that when the entrepreneur abandons a firm of size K_t , the new venture she can run is identical to the one she has abandoned, but the initial capital stock that investors in the new venture are willing to provide is only equal to αK_t . Moreover, when investors in the new venture provide αK_t , they just break even, as stated in (52). Finally, at the moment of contracting at time 0 the entrepreneur has a reservation utility V_0^* , so that the optimal contract must satisfy the participation constraint:

$$V_0 \geq V_0^* . \quad (54)$$

The time-0 participation constraint (54) is always binding under the optimal contract. Otherwise, investors can always increase their payoff by lowering the entrepreneur's consumption and still satisfy all other constraints. However, the entrepreneur's inalienability-of-human-capital constraints (51) are only occasionally binding, as investors dynamically trade off the benefits of providing her with consumption smoothing and the benefits of extracting higher contingent payments from the firm.

We may simplify the contracting problem by summarizing the entire history of the contract via the entrepreneur's promised utility V_t conditional on the history up to time t . Under the optimal contract the dynamics of the agent's promised utility can then be written in the recursive form below. The sum of the agent's utility flow $\zeta U(C_t)dt$ and change in promised utility dV_t has the expected value $\zeta V_t dt$, or:

$$\mathbb{E}_t [\zeta U(C_t)dt + dV_t] = \zeta V_t dt . \quad (55)$$

We can write the stochastic differential equation (SDE) for dV implied by (55) as the sum of: *i*) the expected change (i.e., drift) term $\mathbb{E}_t [dV_t]$; *ii*) a martingale term driven by the Brownian motion \mathcal{Z}_1 ; and *iii*) a martingale term driven by the Brownian motion \mathcal{Z}_2 . Accordingly, we may write the dynamics of the promised utility process V_t as follows:

$$dV_t = \zeta(V_t - U(C_t))dt + x_{1,t}V_t d\mathcal{Z}_{1,t} + x_{2,t}V_t d\mathcal{Z}_{2,t} , \quad (56)$$

where $\{x_{1,t}; t \geq 0\}$ and $\{x_{2,t}; t \geq 0\}$ control the idiosyncratic volatility and systematic volatility of the entrepreneur's promised utility V , respectively.

Finally, we can write the investors' objective as a value function $F(K, V)$ with two state

variables: *i*) the entrepreneur's promised utility V ; and, *ii*) the venture's capital stock K . The optimal contract then specifies investment I , compensation C , idiosyncratic risk exposure x_1 and systematic risk exposure x_2 to maximize the investor's risk-adjusted discounted value of cash flows, as in (50), subject to the entrepreneur's inalienability-of-human-capital constraints (51) and the initial participation constraint (54). Applying Ito's Lemma to $F(K, V)$, we obtain the following HJB equation for investors' value $F(K, V)$:

$$rF(K, V) = \max_{C, I, x_1, x_2} \left\{ (Y - C) + (I - \delta K)F_K + [\zeta(V - U(C)) - x_2\eta V]F_V + \frac{\sigma_K^2 K^2 F_{KK}}{2} + \frac{(x_1^2 + x_2^2)V^2 F_{VV}}{2} + (x_1\epsilon_K + x_2\rho\sigma_K)KV F_{VK} \right\}. \quad (57)$$

To formulate the contract in terms of corporate liquidity and risk exposures it is helpful to express the entrepreneur's promised utility in units of consumption rather than utils. This involves mapping the promised utility V into the promised (certainty-equivalent) wealth W , defined as the solution to the equation $U(bW) = V$, where b is the constant given by (17). We can further reduce the investor's problem to one dimension, with state variable $w = W/K$, by writing the investor's value function $F(K, V)$ as:

$$F(K, V) \equiv F(K, U(bW)) = P(K, W) = p(w) \cdot K, \quad (58)$$

where $p(w)$ is the solution to the ODE provided below in Proposition 5.

Proposition 5 *In the region $w > \underline{w}$, the investors' value $p(w)$ solves:*

$$rp(w) = \max_{i(\cdot)} \left\{ A - i(w) - g(i(w)) + \frac{\chi\gamma}{1-\gamma} (-p'(w))^{1/\gamma} w + (i(w) - \delta)(p(w) - wp'(w)) + \frac{\zeta}{1-\gamma} wp'(w) + \left(\frac{\gamma\sigma_K^2}{2} - \rho\eta\sigma_K \right) \frac{w^2 p'(w) p''(w)}{wp''(w) + \gamma p'(w)} - \frac{\eta^2}{2} \frac{wp'(w)^2}{wp''(w) + \gamma p'(w)} \right\}, \quad (59)$$

subject to the following boundary conditions:

$$\lim_{w \rightarrow \infty} p(w) = q^{FB} - w, \quad (60)$$

$$p(\underline{w}/\alpha) = 0, \quad (61)$$

$$\lim_{w \rightarrow \underline{w}} p''(w) = -\infty. \quad (62)$$

Condition (60) requires that $p(w)$ attains the first best as $w \rightarrow \infty$, and hence, the sum of the entrepreneur's w and the investors' value $p(w)$ equals Tobin's q^{FB} . Condition (61) describes the investors' zero-NPV condition. Finally, Condition (62) is the requirement of equilibrium retention, which involves setting both the idiosyncratic and systematic volatilities of w at \underline{w} to zero. Next, we summarize optimal policy functions.

Proposition 6 *Given $p(w)$, the investment-capital ratio \mathbf{i} , the consumption-capital ratio \mathbf{c} , and risk management policies (x_1, x_2) are respectively given by*

$$g'(\mathbf{i}(w)) = p(w) - wp'(w) - 1, \quad (63)$$

$$\mathbf{c}(w) = \chi(-p'(w))^{1/\gamma} w, \quad (64)$$

$$x_1(w) = \frac{(1 - \gamma)\epsilon_K wp''(w)}{wp''(w) + \gamma p'(w)}, \quad (65)$$

$$x_2(w) = \frac{(1 - \gamma)(\rho\sigma_K wp''(w) + \eta p'(w))}{wp''(w) + \gamma p'(w)}. \quad (66)$$

Equivalence. The optimal liquidity and risk management problem in Section 4 is equivalent to the optimal contracting problem in this section. As illustrated in Table 1, we have

$$s = -p(w) \text{ and } w = m(s), \quad (67)$$

implying $-p \circ m(s) = s$. This equation transparently encapsulates that corporate liquidity is the negative of investors' valuation of the entrepreneur's promised certainty equivalent wealth. Finally, the initial stock of liquidity is $S_0 = s_0 K$ where s_0 is pinned down by the time-0 binding participation constraint given by (54). We provide a proof of the equivalence between the two problems in the Appendix.

6 Quantitative Analysis

In this section, we highlight both qualitative and quantitative implications of our model.

6.1 Parameter Choices and Calibration

While our model is equally tractable for any homogeneous adjustment cost function $g(i)$, for numerical and illustrational simplicity, we choose the following widely-used quadratic adjustment cost function:

$$g(i) = \frac{\theta i^2}{2}, \quad (68)$$

which gives explicit formulae for Tobin's q and optimal i in the first-best MM benchmark:

$$q^{FB} = 1 + \theta i^{FB}, \quad \text{and} \quad i^{FB} = r + \delta - \sqrt{(r + \delta)^2 - 2 \frac{A - (r + \delta)}{\theta}}. \quad (69)$$

For the quadratic adjustment cost function given by (68), the convergence condition (22) for the first best setting implies the following:

$$(r + \delta)^2 - 2 \frac{A - (r + \delta)}{\theta} \geq 0. \quad (70)$$

Our model with constant productivity is parsimonious with eleven parameters. We take the widely used value for the coefficient of relative risk aversion, $\gamma = 2$. We set the equity risk premium to $(\mu_2 - r) = 6\%$ and the annual volatility of the market portfolio return to $\sigma_2 = 20\%$ implying that the Sharpe ratio $\eta = (\mu_2 - r)/\sigma_2 = 30\%$, all standard in the asset pricing literature. We choose the annual risk-free interest rate to be $r = 5\%$ and set the entrepreneur's discount rate equal to the risk-free rate, $\zeta = r = 5\%$.

For the production-side parameters, we use estimates in Eberly, Rebelo, and Vincent (2009) and set the annual productivity A at 20% and the annual volatility of capital shocks at $\sigma_K = 20\%$. We set the correlation between the market portfolio return and the depreciation shock $\rho = 0.2$, which implies that the idiosyncratic volatility of the depreciation shock is $\epsilon_K = 19.6\%$. We fit the first-best values of q^{FB} and i^{FB} to the sample averages by setting the adjustment cost parameter at $\theta = 2$ and the (expected) annual capital depreciation rate at $\delta_K = 11\%$ both of which are in line with estimates in Hall (2004) and Riddick and Whited (2009). These parameters imply that $q^{FB} = 1.264$, $i^{FB} = 0.132$, and $\beta^{FB} = 0.2$. Finally, we let $\alpha = 0.8$, in line with estimates reported in Li, Whited, and Wu (2016). The parameter values for our baseline case are summarized in Table 2.

Table 2: PARAMETER VALUES

This table summarizes the parameter values for our baseline model with no productivity shocks. Whenever applicable, parameter values are annualized.

Parameters	Symbol	Value
Risk-free rate	r	5%
The entrepreneur's discount rate	ζ	5%
Correlation	ρ	20%
Excess market portfolio return	$\mu_2 - r$	6%
Volatility of market portfolio	σ_2	20%
The entrepreneur's relative risk Aversion	γ	2
Capital depreciation rate	δ_K	11%
Volatility of capital depreciation shock	σ_K	20%
Quadratic adjustment cost parameter	θ	2
Firm's productivity	A	20%
Inalienability-of-human-capital parameter	α	80%

6.2 Investors' Value is the Entrepreneur's Liability: $p(w) = -s$

We begin by plotting the solution for respectively $p(w)$ and $m(s)$.

Promised Wealth w and Investors' Value $p(w)$. Panels A and B of Figure 1 plot $p(w)$ and $p'(w)$. Under the first best, compensation to the entrepreneur is simply a one-to-one transfer away from investors, as we see from the dotted lines: $p(w) = q^{FB} - w = 1.264 - w$ and $p'(w) = -1$. Under inalienability of human capital, $p(w)$ is decreasing and concave in w . That is, as w increases the entrepreneur is less constrained. In the limit, as $w \rightarrow \infty$, $p(w)$ approaches $q^{FB} - w$, and $p'(w) \rightarrow -1$, so that the first-best payoff obtains when the entrepreneur is unconstrained. However, the entrepreneur's inability to fully commit not to walk away *ex post* imposes a lower bound \underline{w} on w . For our parameter values, $w \geq \underline{w} = 0.944$. That is, the entrepreneur receives at least 94.4% in promised certainty equivalent wealth for each unit of capital stock, which is greater than $\alpha = 0.8$.

Finally, note that despite being fully diversified, investors behave in an under-diversified manner due to the entrepreneur's inalienability-of-human-capital constraints. This is reflected in the concavity of investors' certainty equivalent value function $p(w)$. This concav-

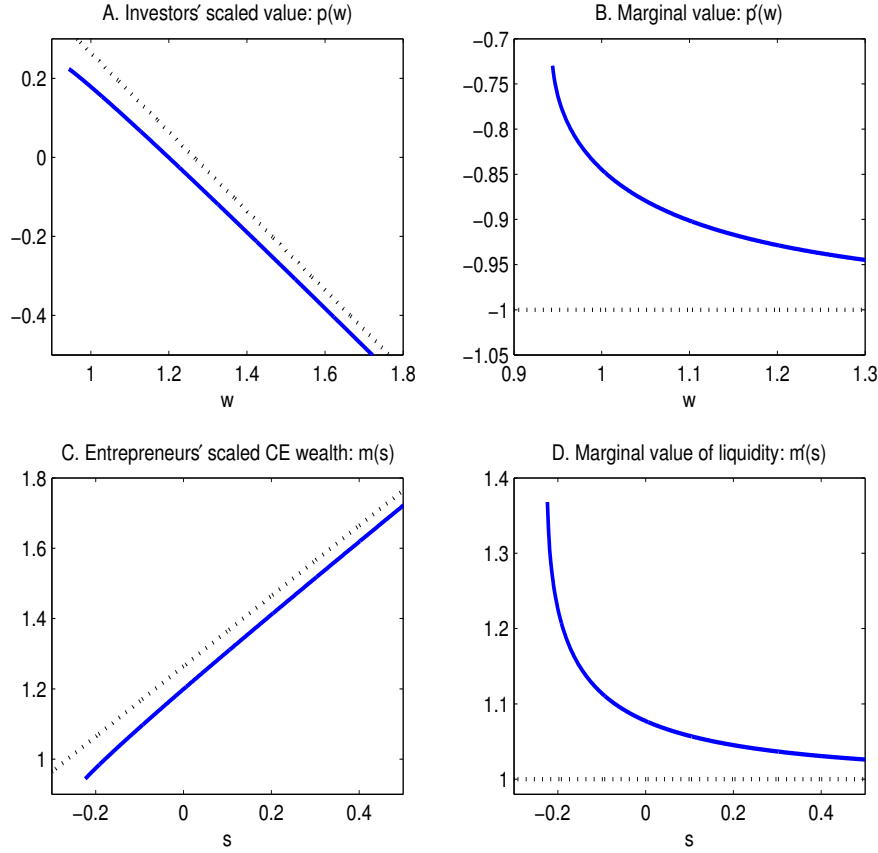


Figure 1: **Investors' value $p(w)$, marginal value $p'(w)$, the entrepreneur's certainty equivalent wealth $m(s)$ and marginal value of liquidity $m'(s)$.** For the limited-commitment case, $w \geq \underline{w} = 0.944$, and $p(w)$ is decreasing and concave. Equivalently, $s \geq \underline{s} = -0.224$, and $m(s)$ is increasing and concave. The dotted lines depict the First-Best results: $p(w) = q^{FB} - w$, $p'(w) = -1$, $m(s) = q^{FB} + s$, and $m'(s) = 1$ with $q^{FB} = 1.264$.

ity property is an important difference of the limited commitment problem compared to the first-best benchmark. Next, we turn to the solution for the entrepreneur's liquidity and risk management problem.

Liquidity s and the Entrepreneur's Certainty-Equivalent Wealth $m(s)$. Panels C and D of Figure 1 plot $m(s)$ and the marginal value of liquidity $m'(s)$. As one may expect $m(s)$ is increasing and concave in s . The higher the liquidity s the less constrained is the entrepreneur, so that $m'(s)$ decreases ($m''(s) < 0$). In the limited-commitment case, $m(s)$ approaches $q^{FB} + s$ and $m'(s) \rightarrow 1$ as $s \rightarrow \infty$. The entrepreneur's LOC limit is $\underline{s} = -p(\underline{w}) = -p(0.944) = -0.224$.

Note that $(s, m(s))$ is the “mirror-image” of $(-p(w), w)$. To be precise, rotating Panel A counter-clock-wise by 90° (by turning the original x -axis (i.e., w) into the new y -axis $m(s)$) and adding a minus sign to the horizontal x -axis (i.e., $-p(w) = s$ is the new x -axis) produce Panel C.

6.3 Investment, Consumption, Liquidity, and Risk Management

We first analyze the firm’s investment decisions, then the entrepreneur’s optimal consumption, and finally the idiosyncratic and systematic risk-exposure policies.

Investment, Marginal Private q , and Marginal (Private) Value of Liquidity $m'(s)$.

The FOC for investment is:

$$1 + g'(i(s)) = \frac{J_K}{J_S} = \frac{M_K}{M_S} = \frac{m(s) - sm'(s)}{m'(s)}, \quad (71)$$

where the second equality follows from the definition of the value function in (35), and the last equality follows from the homogeneity property of $M(K, S)$ in K . Under perfect capital markets the entrepreneur’s certainty equivalent wealth is given by $M(K, S) = m(s) \cdot K = (q^{FB} + s) \cdot K$ and the marginal value of liquidity is $M_S = 1$ at all times. Hence in this case, the FOC (71) specializes to the classical Hayashi condition for optimal investment, where the marginal cost of investing $1 + g'(i(s))$ equals marginal q .

Under limited commitment, $M_S > 1$ in general and the FOC (71) then states that the marginal cost of investing equals the ratio between (a) M_K , the marginal private q , and (b) M_S , the marginal private value of liquidity. Unlike in the classical q theory of investment, financing is costly as reflected by $M_S > 1$.

Figure 2 illustrates the effect of inalienability of human capital on investment $i(s)$ and marginal private q . The dotted lines in Panels A and B of Figure 2 give the first-best $i^{FB} = 0.132$ and $q^{FB} = 1.264$, respectively. With limited commitment, $i(s)$ is lower than the first-best benchmark $i^{FB} = 0.132$ for all s , and increases from -0.043 to $i^{FB} = 0.132$ as s increases from the left boundary $\underline{s} = -0.224$ towards ∞ . This is to be expected: increasing financial slack mitigates the severity of under-investment for a financially constrained firm. When $S > 0$, marginal private q , $M_K = m(s) - sm'(s)$, increases with s , which is consistent

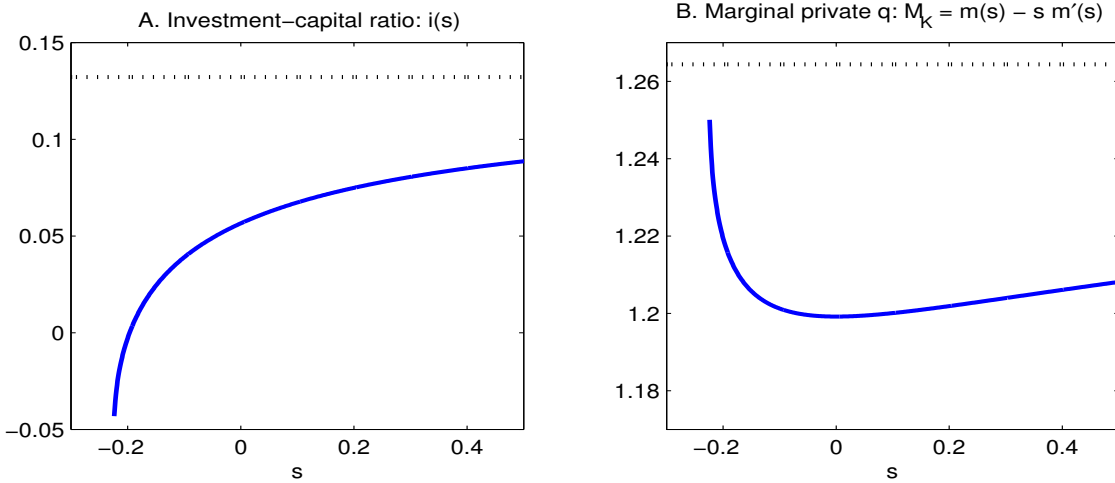


Figure 2: **The investment-capital ratio $i(s)$, and marginal private q , $M_K = m(s) - sm'(s)$.** For the limited-commitment case, the firm always under-invests and $i(s)$ increases with s . The dotted line depicts the full-commitment MM results where the marginal equals $q^{FB} = 1.264$ and the first-best investment-capital ratio $i^{FB} = 0.132$.

with our intuition. The higher the level of liquidity s , the higher is marginal private q . Note however that, surprisingly, M_K *decreases* with s from 1.25 to 1.20 in the credit region $s < 0$. What is the intuition? When the firm is financing its investment via credit at the margin (when $S < 0$), increasing K moves a negative-valued s closer to the origin thus mitigating financial constraints, which is an additional benefit of accumulating capital.¹⁵

But why does a high marginal- q firm invest less in the credit region $s < 0$? And how do we reconcile an increasing investment function $i(s)$ with a decreasing marginal q function, $M_K = m(s) - sm'(s)$ in the credit region $s < 0$? The reason is simply that in the credit region ($s < 0$) a high marginal- q firm also has a high $m'(s)$ reflecting a high marginal financing cost. When $s < 0$, marginal q and marginal financing cost $m'(s)$ are perfectly correlated. And investment is determined by the ratio between the marginal q and $m'(s)$ as we have noted. At the left boundary $\underline{s} = -0.224$ marginal q is 1.25 and $m'(s)$ is 1.37, both of which are high. Together they imply that it is optimal to engage in asset sales, as $i(-0.224) = ((1.25/1.37) - 1)/2 = -0.043 < 0$, which is significantly lower than the first-best $i^{FB} = 0.132$.

Taking the derivative of $i(s)$ in (71) with respect to s , we find that the investment-cash

¹⁵Formally, this result follows from $dM_K/ds = -sm''(s) < 0$ when $s < 0$ and from the concavity of $m(s)$.

sensitivity is positive as

$$i'(s) = -\frac{1}{\theta} \frac{m(s)m''(s)}{m'(s)^2} > 0. \quad (72)$$

This result follows from the concavity of $m(s)$ both when $s \geq 0$ and $s < 0$, and explains why investment is increasing in s as shown in Panel A.

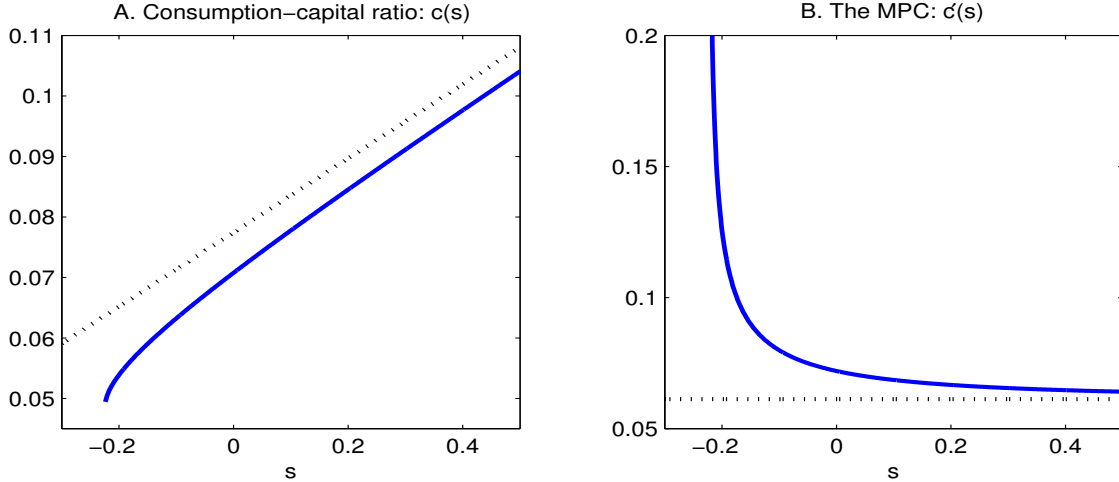


Figure 3: **Consumption-capital ratio $c(s)$ and the MPC $c'(s)$.** For the limited-commitment case, $c(s)$ is increasing and concave in s , but is lower than the First-Best benchmark $c^{FB}(s)$. The dotted lines depict the First-Best results: $c(s) = \chi(s + q^{FB})$ and the MPC $c'(s) = \chi = 6.13\%$ with $q^{FB} = 1.264$.

Optimal Consumption and the MPC. Figure 3 plots consumption $c(s)$, and the MPC $c'(s)$ in Panels A and B respectively. The dotted lines give the first-best results: $c(s) = \chi(s + q^{FB})$ and MPC $c'(s) = 6.13\%$. The solid line gives the entrepreneur's consumption, which is lower than the first-best benchmark. As one might expect, the higher the financial slack s the higher is $c(s)$ as seen in the figure. Moreover, we have $m(s) \rightarrow q^{FB} + s$ and the marginal value of liquidity $m'(s) \rightarrow 1$ as $s \rightarrow \infty$, so that $c(s) \rightarrow \chi(q^{FB} + s)$, the permanent-income consumption benchmark. Panel B shows that the MPC $c'(s)$ decreases significantly with s and approaches the permanent-income benchmark $\chi = 6.13\%$ as $s \rightarrow \infty$. Thus, financially constrained entrepreneurs deep in debt (with s close to \underline{s}) have MPCs that are substantially higher than the permanent-income benchmark, consistent with empirical evidence documented by Parker (1999) and Souleles (1999). Next we turn to the firm's idiosyncratic and systematic risk exposures.

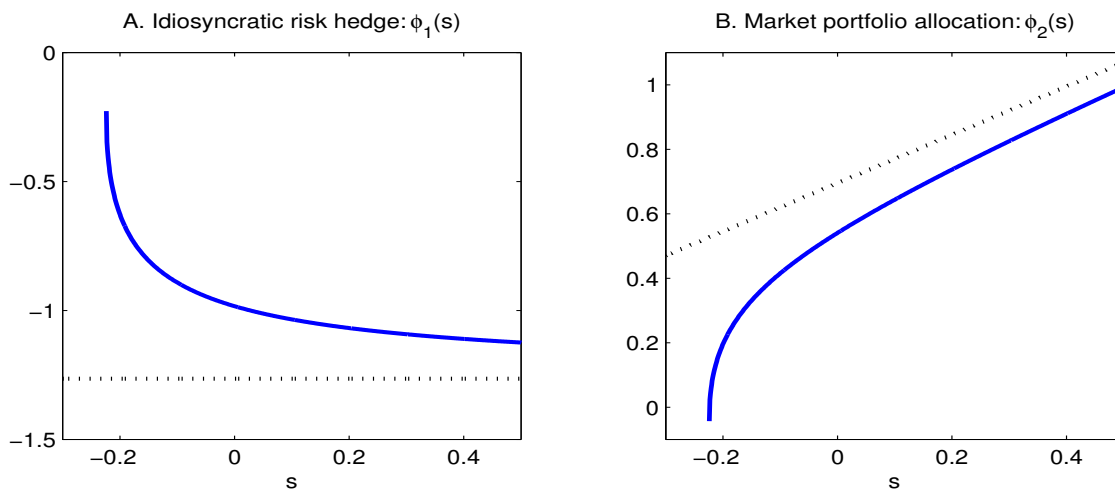


Figure 4: **Idiosyncratic risk hedge $\phi_1(s)$ and market portfolio allocation $\phi_2(s)$.** For the limited-commitment case, the idiosyncratic risk hedge $\phi_1(s) < 0$, its absolute size $|\phi_1(s)|$ is increasing and concave, and is lower than the First-Best result $|\phi_1^{FB}(s)|$. The market portfolio allocation $\phi_2(s)$ is also increasing and concave, and lower than the First-Best result $\phi_2^{FB}(s)$. The dotted lines depict the First-Best results: $\phi_1^{FB}(s) = -q^{FB}$ and $\phi_2^{FB}(s) = -0.2 \times q^{FB} + 0.75 \times (s + q^{FB})$ where $q^{FB} = 1.264$.

Dynamic Risk Management and Market Risk Exposures. Panel A of Figure 4 plots the hedging demand against idiosyncratic shocks $\phi_1(s)$. Under the first best, the entrepreneur is fully insured against idiosyncratic business risk, by taking a short position $\phi_1(s) = -q^{FB} = -1.264$ perfectly offsetting the entrepreneur's long position in the risky venture. However, with limited commitment the entrepreneur cannot fully hedge her idiosyncratic risk exposure. How does $\phi_1(s)$ depend on s in this case? As the firm becomes more constrained (s decreases) the entrepreneur decreases her hedging demand $|\phi_1(s)|$. In the limit, when $s = \underline{s}$, the entrepreneur must turn off the idiosyncratic volatility by setting $\phi_1(\underline{s}) = \underline{s} = -0.224$ in order to survive. In the other direction, as $s \rightarrow \infty$ the entrepreneur can fully diversify the idiosyncratic business risk by setting $\lim_{s \rightarrow \infty} \phi_1(s) = -q^{FB} = -1.264$, attaining the First-Best perfect-insurance value-maximizing benchmark. In simple economic terms, when s is low, risk management is about guaranteeing the survival of the firm by turning off the volatility of s , while when s is large, the firm's idiosyncratic risk management is all about eliminating exposure to idiosyncratic risk. Overall, a less constrained firm has a larger hedging position $|\phi_1(s)|$ (even after controlling for firm size.) Rampini, Sufi, and

Viswanathan (2014) provide empirical evidence supporting this result.¹⁶ Note that here liquidity s and hedging $\phi_1(s)$ are complements.

Panel B plots the market portfolio position $\phi_2(s)$. Under the first best, the entrepreneur wants to invest like a fully-diversified mean-variance investor, as in Merton (1971). How does she achieve this goal given her significant non-diversifiable exposure to her business? In addition to completely off-loading her idiosyncratic risk exposure by setting $\phi_1^{FB}(s) = -q^{FB}$, she also effectively “sells” her business at the market price q^{FB} per unit of capital by taking a short position $-\beta^{FB}q^{FB}$ in the market portfolio. Finally, she behaves as a standard Merton investor with total net worth of $S_t + q^{FB}K_t$, who optimally allocates a fixed fraction, $\eta/(\gamma\sigma_2)$, of her entire net worth in the market portfolio.

With limited commitment, the entrepreneur cannot achieve the desired first-best exposure to the market portfolio for this would result in excessively volatile liquidity s . Indeed, when $s = \underline{s}$, the entrepreneur must also turn off the systematic volatility by setting $\phi_2(\underline{s}) = \beta^{FB}\underline{s} = -0.045$ in order to survive. More generally, for a highly constrained firm, much of risk management is about survival. In contrast, as $s \rightarrow \infty$ the firm is flush with liquidity and the entrepreneur behaves as a mean-variance Merton investor by letting $\phi_2(s) \rightarrow \phi_2^{FB}(s)$.

7 Two-Sided Limited Commitment

Under only one-sided limited commitment, the optimal policy is such that investors may incur losses when w is large. As Figure 1 illustrates, $p(w) < 0$ when $w > 1.18$. To be able to retain the entrepreneur, investors then promise such a high w to the entrepreneur that they end up committing to making losses in these states of the world. But, what if they cannot commit to such loss-making promises to the entrepreneur? We explore this issue in this section and characterize the solution when neither the entrepreneur nor investors are able to commit.

The main change relative to the one-sided commitment problem is that the upper boundary is now $\bar{s} = -p(\bar{w}) = 0$, since any promise of strictly positive savings $s > 0$ is not credible, as this involves a negative continuation value for investors. As it turns out, solving the two-

¹⁶Li, Whited, and Wu (2016) structurally estimate a model featuring taxes and limited commitment (along the lines of Rampini and Viswanathan (2013)), and provide empirical evidence in support of a limited-commitment-based collateral mechanism.

sided limited commitment problem does not involve major additional complexities. It implies the following conditions at the new upper boundary $\bar{s} = 0$:

$$\sigma_1^s(0) = \sigma_2^s(0) = 0. \quad (73)$$

Using the same argument as for (47), we may equivalently express (73) as

$$\lim_{s \rightarrow 0} m''(s) = -\infty. \quad (74)$$

Moreover, we need to verify that the drift $\mu^s(s)$ given in (42) is weakly negative at $\bar{s} = 0$, so that liquidity s is weakly decreasing with probability one at $\bar{s} = 0$.

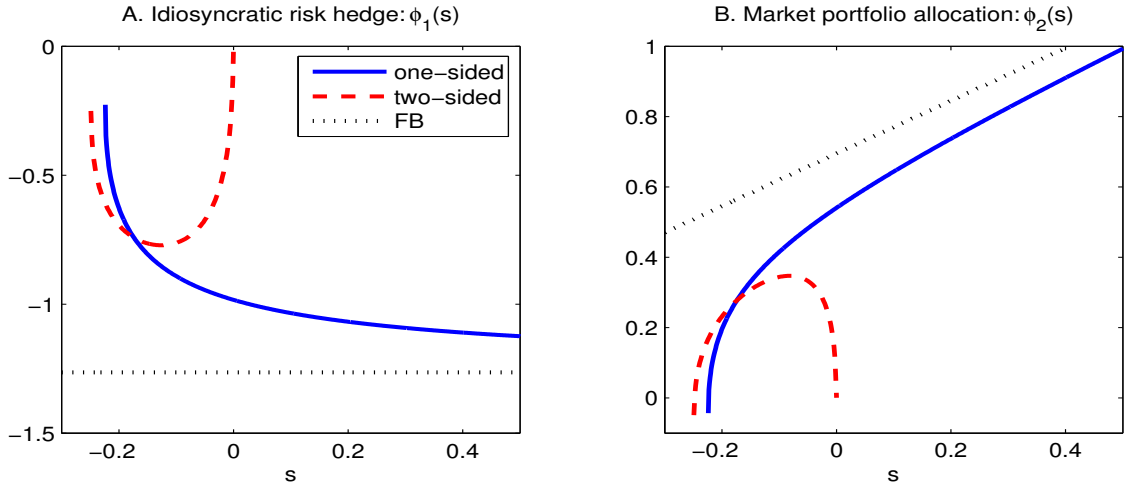


Figure 5: **Idiosyncratic risk hedge $\phi_1(s)$ and market portfolio allocation $\phi_2(s)$ under two-sided limited-commitment case.** The endogenous upper boundary $\bar{s} = 0$. For the two-sided limited-commitment case, neither $\phi_1(s)$ nor $\phi_2(s)$ is monotonic in s .

Liquidity Buffer and Risk Management. Figure 5 plots the idiosyncratic risk hedge position $\phi_1(s)$ and the market portfolio allocation $\phi_2(s)$ for both one-sided and two-sided limited commitment cases. It shows that in the two-sided limited-commitment case, s lies between $\underline{s} = -0.25$ and $\bar{s} = 0$, so that the entrepreneur has a larger LOC limit of $|\underline{s}| = 0.25$. But a higher LOC limit $|\underline{s}|$ comes with a lower promised utility. Essentially, the additional investor limited-liability condition limits the entrepreneur's self-insurance capacity. Remarkably, under two-sided limited commitment the LOC limit is larger. In other words, a firm

with a larger debt capacity is not necessarily a less constrained firm. Moreover, it may have a lower value!

Figure 5 illustrates that both $\phi_1(s)$ and $\phi_2(s)$ are non-monotonic in s in the two-sided limited-commitment case. The reason is that the volatilities $\sigma_1^s(s)$ and $\sigma_2^s(s)$ must be turned off at both $\underline{s} = -0.25$ and $s = 0$ to prevent the entrepreneur and investors from separating. This is achieved by setting, respectively, $\phi_1(\underline{s}) = \underline{s} = -0.25$, $\phi_2(\underline{s}) = \beta^{FB} \underline{s} = -0.05$, and $\phi_1(0) = 0$, $\phi_2(0) = 0$, as implied by the volatility boundary conditions for $\sigma_1^s(s)$ and $\sigma_2^s(s)$ at both boundaries.

Investment $i(s)$ and Entrepreneur's Certainty Equivalent Wealth $m(s)$. Panel A of Figure 6 reports the two-sided limited-commitment solution for investment. Compared with the First-Best benchmark, the firm under-invests when $s < -0.13$, but over-invests when $-0.13 < s \leq 0$. Whether it under-invests or over-invests depends on the net effects of the entrepreneur's limited-commitment and investors' limited-liability constraints. For sufficiently low values of s (when the entrepreneur is deep in debt) the entrepreneur's constraint matters more and hence the firm under-invests. When s is sufficiently close to zero, investors' limited-liability constraint has a stronger influence on investment. To ensure that s will not grow the entrepreneur needs to transform liquid into illiquid savings. This causes the firm to over-invest relative to the first-best.

Phrased in terms of the equivalent optimal contracting problem, the intuition is as follows. Given that the entrepreneur cares about the total compensation $W = w \cdot K$ and given that investors are constrained by their ability to promise the entrepreneur w beyond an upper bound \bar{w} , (in this case, $\bar{w} = m(0) = 0.843$), investors reward the entrepreneur along the extensive margin, firm size K , which induces over-investment but allows the entrepreneur to build more human capital.

Panel B of Figure 6 plots the entrepreneur's certainty equivalent wealth $m(s)$. As one would expect, $m(s)$ increases with s , but $m(s)$ for the two-sided case is much lower than that for the one-sided case, as the additional constraints on the investors' side make liquidity costly and lower the total surplus.

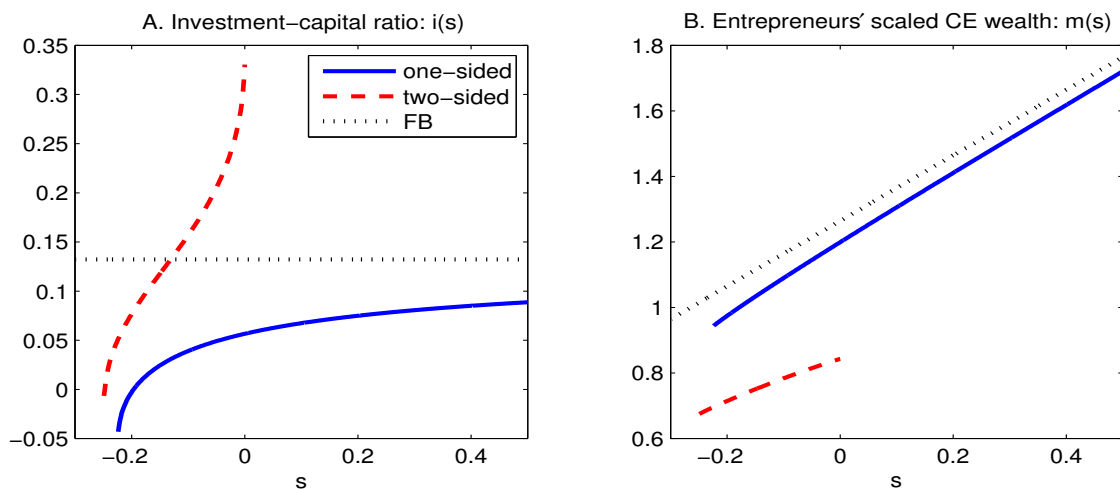


Figure 6: **Optimal investment-capital ratio $i(s)$ and the entrepreneur's certainty equivalent wealth $m(s)$ under the two-sided limited-commitment case.** Compared with the First-Best benchmark, for the two-sided limited-commitment case, liquidity s lies in the range $(\underline{s}, \bar{s}) = (-0.249, 0)$, and the firm under-invests when s is sufficiently close to $\underline{s} = -0.249$ and over-invests when s is sufficiently close to $\bar{s} = 0$. The firm's LOC limit for the two-sided limited-commitment case $|\underline{s}| = 0.249$ is larger than the LOC limit $|\underline{s}| = 0.224$ for the one-sided limited-commitment case, but the firm cannot save for the two-sided limited-commitment case.

8 Persistent Productivity Shocks

In this section, we extend the model by introducing persistent productivity shocks that have first-order implications for corporate liquidity and risk management. The firm faces two conflicting forces in the presence of such shocks. First, as Froot, Scharfstein and Stein (1993) and Rampini and Viswanathan (2010) have emphasized, the firm will want to make sure that it has sufficient liquidity and funding capacity to be able to take full advantage of the investment opportunities that become available when productivity is high. To do so, the firm may want to take hedging positions that allow it to transfer funds from the low to the high productivity state. Second, the firm also wants to smooth the entrepreneur's consumption across productivity states, allowing the entrepreneur to consume a higher share of earnings in the low than in the high productivity state. To do so, the firm will need to ensure that it has sufficient liquidity and funding capacity in the low productivity state. This may require taking hedging positions such that funds are transferred from the high to the low productivity state.

Which of these two forces dominates? We show that even for extreme parameter values for the productivity shocks the consumption smoothing effect dominates. Part of the reason is that, when productivity is high, the firm's endogenous credit limit is also high, so that transferring funds from the low to the high productivity state is not that important. In contrast, the consumption smoothing benefits of transferring funds from the high to the low productivity state are significant.

Without much loss of generality we model persistent productivity shocks $\{A_t; t \geq 0\}$ as a two-state Markov switching process, $A_t \in \{A^L, A^H\}$ with $0 < A^L < A^H$. We denote by $\lambda_t \in \{\lambda^L, \lambda^H\}$ the transition intensity from one state to the other, with λ^L denoting the intensity from state L to H , and λ^H the intensity from state H to L . The counting process $\{\mathcal{N}_t; t \geq 0\}$ (starting with $\mathcal{N}_0 = 0$) keeps track of the number of times the firm has switched productivity state up to time t ; it increases by one whenever the state switches from either H to L or from L to H : $d\mathcal{N}_t = \mathcal{N}_t - \mathcal{N}_{t-} = 1$ if and only if $A_t \neq A_{t-}$, otherwise, $d\mathcal{N}_t = 0$.

In the presence of such shocks the entrepreneur will want to purchase or sell insurance against stochastic changes in productivity. We characterize the optimal insurance policy against such shocks and how investment, consumption, risk management, and the firm's credit limit vary with the firm's productivity. For brevity, we only consider the one-sided limited-commitment case where productivity shocks are purely idiosyncratic.¹⁷

Productivity Insurance Contract. Consider the following insurance contract offered at current time $t-$. Over the time interval $dt = (t-, t)$, the entrepreneur pays the unit insurance premium $\xi_{t-}dt$ to the insurance counterparty in exchange for a unit payment at time t if and only if $A_t \neq A_{t-}$ (i.e., $d\mathcal{N}_t = 1$). That is, the underlying event for this insurance contract is the change in productivity. Under our assumptions of perfectly competitive financial markets and idiosyncratic productivity shocks, the actuarially fair insurance premium is given by the intensity of the change in productivity state: $\xi_{t-} = \lambda_{t-}$.¹⁸

Let Π_{t-} denote the number of units of insurance purchased by the entrepreneur at time $t-$. We refer to Π_{t-} as the insurance demand. If $\Pi_{t-} < 0$, the firm sells insurance and

¹⁷We have analyzed more general situations that incorporate systematic productivity shocks and two-sided limited-commitment. The generalized liquidity and risk management problem in this section also has an equivalent optimal contracting formulation. This material is available upon request.

¹⁸We can generalize the model to allow for a systematic risk premium. This requires using the standard change of measure technique of choosing different jump intensities under the physical measure and the risk-neutral measure. Results are available upon request.

collects insurance premia at the rate of $\lambda_{t-}\Pi_{t-}$. Liquidity S_t then accumulates according to:

$$dS_t = (rS_t + Y_t - C_t + \Phi_{2,t}(\mu_2 - r) - \lambda_{t-}\Pi_{t-}) dt + \Phi_{1,t}\epsilon_K d\mathcal{Z}_{1,t} + \Phi_{2,t}\sigma_2 d\mathcal{Z}_{2,t} + \Pi_{t-} d\mathcal{N}_t. \quad (75)$$

The first term in (75) is the expected rate of return, net of the insurance premium payment $\lambda_{t-}\Pi_{t-}$, from investing in respectively, the risk-free asset and the two risky financial assets. The second and third terms are the standard diffusion terms associated with the two risky financial assets, and the last term is the insurance payment contingent on the change in productivity state.

We write the solution for the firm's value as a pair of state-contingent value functions $J(K, S; A^L) \equiv J^L(K, S)$ and $J(K, S; A^H) \equiv J^H(K, S)$, which solve two inter-linked HJB equations, one for each state.¹⁹ The HJB equation in state L is thus:²⁰

$$\begin{aligned} \zeta J^L(K, S) = \max_{C, I, \Phi_1, \Phi_2, \Pi^L} & \quad \zeta U(C) + (I - \delta_K K) J_K^L + \frac{\sigma_K^2 K^2}{2} J_{KK}^L \\ & + (rS + \Phi_2(\mu_2 - r) + A^L K - I - G(I, K) - C - \lambda^L \Pi^L) J_S^L \\ & + (\epsilon_K^2 \Phi_1 + \rho \sigma_K \sigma_2 \Phi_2) K J_{KS}^L + \frac{(\epsilon_K \Phi_1)^2 + (\sigma_2 \Phi_2)^2}{2} J_{SS}^L \\ & + \lambda^L [J^H(K, S + \Pi^L) - J^L(K, S)]. \end{aligned} \quad (76)$$

Two important features differentiate (76) from the HJB equation (11) in the baseline case. First, the drift term involving the marginal utility of liquidity J_S^L now includes the insurance payment $-\lambda^L \Pi^L$. Second, the last term in (76) captures the endogenous adjustment of S by the amount of Π^L and the corresponding change in the value function following a productivity change from A^L to A^H .

The inalienability-of-human capital constraint must also hold at all time t in both productivity states, so that

$$S_t \geq \underline{S}(K_t; A_t), \quad (77)$$

or equivalently,

$$s_t \geq \underline{s}(A_t). \quad (78)$$

¹⁹For contracting models involving jumps and/or regime switching, see Biais, Mariotti, Rochet, and Villeneuve (2010), Piskorski and Tchistyi (2010), and DeMarzo, Fishman, He and Wang (2012), among others.

²⁰For brevity, we omit the coupled equivalent HJB equation for $J(K, S; A^H) \equiv J^H(K, S)$ in state H .

Naturally, the firm's time- t endogenous credit limit $|\underline{s}(A_t)|$ depends on its productivity A_t .

The entrepreneur determines her optimal insurance demand Π^L in state L by differentiating (76) with respect to Π^L and setting Π^L to satisfy the FOC:

$$J_S^L(K, S) = J_S^H(K, S + \Pi^L), \quad (79)$$

provided that the solution Π^L to the above FOC satisfies the (state-contingent) condition:

$$S + \Pi^L \geq \underline{S}^H. \quad (80)$$

Otherwise, the entrepreneur sets the insurance demand so that $\Pi^L = \underline{S}^H - S$, in which case the firm will be at its maximum debt level \underline{S}^H when productivity switches from A^L to A^H .²¹

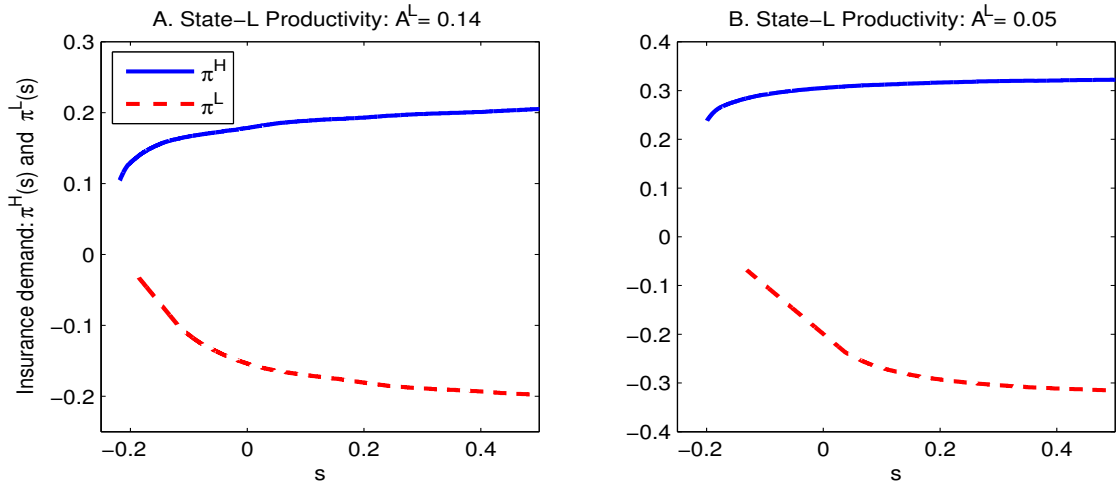


Figure 7: **Insurance demand:** $\pi^H(s)$ and $\pi^L(s)$. State- H productivity is $A^H = 0.25$ in both panels. In Panel A, State- L productivity is $A^L = 0.14$ and $\pi^L(s) = \underline{s}^H - s$ when $-0.186 < s < -0.114$. In Panel B, State- L productivity is $A^L = 0.05$ and $\pi^L(s) = \underline{s}^H - s$ when $-0.131 < s < 0.039$.

Quantitative Analysis. We consider two sets of parameter values. The first set is such that $A^H = 0.25$, $A^L = 0.14$, and $\lambda^L = \lambda^H = 0.2$, with all other parameter values as in Table 2. The transition intensities $(\lambda^H, \lambda^L) = (0.2, 0.2)$ imply that the expected duration of each

²¹There is an equivalent set of conditions characterizing Π^H in state H , which we omit.

state is five years. The second set of parameter values is identical to the first, except that $A^L = 0.05$. That is, productivity in the low state, A^L , is much lower (0.05 instead of 0.14).

Figure 7 plots the entrepreneur's insurance demand $\pi^H(s)$ as the solid line, and $\pi^L(s)$ as the dashed line. We use \underline{s}^H and \underline{s}^L to denote $\underline{s}(A_t)$ when $A_t = A^H$ and $A_t = A^L$, respectively. Panel A plots the insurance demand in both states when productivity differences are $(A^H - A^L)/A^H = (0.25 - 0.14)/0.25 = 44\%$, while Panel B plots the insurance demand when productivity differences are very large, $(A^H - A^L)/A^H = (0.25 - 0.05)/0.25 = 80\%$. Remarkably, under both sets of parameter values the firm optimally buys insurance in state H , $\pi^H(s) > 0$, and sells insurance in state L , $\pi^L(s) < 0$. This result is not obvious *a priori*, for when productivity differences are large the benefit from transferring liquidity from state L to H and thereby taking better advantage of investment opportunities when they arise, could well be the dominant consideration for the firm's risk management. But that turns out not to be the case. Even when productivity differences are as large as 80%, the dominant consideration is still to smooth the entrepreneur's consumption. Moreover, a comparison of Panels A and B reveals that for the larger productivity differences, the insurance demand is also larger, with $\pi^H(s)$ exceeding 0.2 everywhere in Panel B, but remaining well below 0.2 in Panel A, and $\pi^L(s)$ attaining values lower than -0.25 in Panel B (when $s + \pi^L \geq \underline{s}^H$ is not binding), while $\pi^L(s)$ is always larger than -0.2 in Panel A.²²

9 Conclusion

The theory of corporate liquidity and risk management we have developed is particularly relevant for industries where an essential input is the human capital and talent of their employees, e.g., information technology. The survival of these companies rests on their ability to retain talent. We have shown that the optimal way for the firm to retain talent is by offering both current pay and future promised compensation. But the promised future compensation must be credible, which means that the firm must back these promises with corporate savings, or liquidity. This is the main motivation for the firm's liquidity management in our model which we formalize this insight via an optimal contracting formulation. Importantly, it is not sufficient for the firm to back up promises with physical capital accumulation alone,

²²These results are robust and hold for other more extreme parameter values, which for brevity we do not report.

because physical capital while productive is illiquid and costly to unwind. Although more physical capital does expand the firm’s borrowing capacity, by extending its line of credit limit, it is suboptimal for the firm to fully draw down its line of credit, for then the firm’s ability to honor its future compensation promises is compromised. In addition, corporate risk management complements the firm’s liquidity management by reducing unnecessary exposures of the firm’s savings to idiosyncratic risk but also capturing risk-adjusted excess returns on behalf of key employees and thereby enhancing the firm’s ability to make credible future promises.

Our model also generalizes existing theories of corporate investment, financing, liquidity and risk management, by adding optimal compensation of the entrepreneur to the stew. Our analysis, thus, contributes to the literature on executive compensation, which typically abstracts from financial constraints (see Frydman and Jenter, 2010, and Edmans and Gabaix, 2016, for recent surveys). Our analysis brings out an important positive link between compensation and corporate liquidity, and helps explain why companies typically cut compensation, scale back investment, and reduce risk management positions, when liquidity is tight. It also explains why companies simultaneously sell insurance. What has generally been interpreted as a form of gambling for resurrection—selling insurance by financially constrained firms—can be understood under our analysis as an efficient attempt by the firm to relax its financial constraints.

For when liquidity is tight (when the firm is close to exhausting its line of credit), the priority for the firm is to survive. From a liquidity and risk management perspective, this means that the firm cuts back expenditure (e.g., compensation and investment), generates new sources of liquidity by selling insurance on persistent productivity shocks and taking hedging positions to ensure that the volatility of its liquid savings s is minimal. In contrast, when liquidity is plentiful the firm’s financial policy switches to maximizing the present value of its liquid and illiquid capital, $(s + q^{FB})K$, provided that investors can credibly commit to accumulating future promises. When neither the entrepreneur nor investors can commit, i.e., for the two-sided limited commitment problem, we show that it is optimal for the firm to distort its investment, and over-invest in physical capital when it cannot credibly accumulate more liquid savings.

Although our framework is already quite rich, we have imposed a number of strong assumptions, which are worth relaxing in future work. For example, one interesting direction

is to allow for equilibrium separation between the entrepreneur and investors. This could arise, when after an adverse productivity shock the entrepreneur no longer offers the best use of the capital stock. Investors may then want to redeploy their capital to other more efficient uses. By allowing for equilibrium separation our model could be applied to study questions such as the expected and optimal life-span of entrepreneurial firms, the optimal turnover of managers, or the optimal investment in firm-specific or general human capital.

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Appendix

A The Entrepreneur's Optimization Problem

We conjecture that the value function $J(K, S)$ takes the following form:

$$J(K, S) = \frac{(bM(K, S))^{1-\gamma}}{1-\gamma} = \frac{(bm(s)K)^{1-\gamma}}{1-\gamma}, \quad (\text{A.1})$$

where b is given in (17). We then have:

$$J_S = b^{1-\gamma}(m(s)K)^{-\gamma}m'(s), \quad (\text{A.2})$$

$$J_K = b^{1-\gamma}(m(s)K)^{-\gamma}(m(s) - sm'(s)), \quad (\text{A.3})$$

$$J_{SK} = b^{1-\gamma}(m(s)K)^{-1-\gamma}(-sm(s)m''(s) - \gamma m'(s)(m(s) - sm'(s))), \quad (\text{A.4})$$

$$J_{SS} = b^{1-\gamma}(m(s)K)^{-1-\gamma}(m(s)m''(s) - \gamma m'(s)^2), \quad (\text{A.5})$$

$$J_{KK} = b^{1-\gamma}(m(s)K)^{-1-\gamma}(s^2m(s)m''(s) - \gamma(m(s) - sm'(s))^2). \quad (\text{A.6})$$

Substituting these terms into the HJB equation (11) and simplifying, we obtain:

$$\begin{aligned} 0 = & \max_{c, i, \phi_1, \phi_2} \zeta m(s) \frac{\left(\frac{c}{bm(s)}\right)^{1-\gamma} - 1}{1-\gamma} + (i - \delta_K)(m(s) - sm'(s)) \\ & + (rs + \phi_2(\mu_2 - r) + A - i - g(i) - c)m'(s) + \frac{\sigma_K^2}{2} \left(s^2 m''(s) - \frac{\gamma(m(s) - sm'(s))^2}{m(s)} \right) \\ & + (\epsilon_K^2 \phi_1 + \rho \sigma_K \sigma_2 \phi_2) \left(-sm''(s) - \frac{\gamma m'(s)(m(s) - sm'(s))}{m(s)} \right) \\ & + \frac{(\epsilon_K \phi_1)^2 + (\sigma_2 \phi_2)^2}{2} \left(m''(s) - \frac{\gamma m'(s)^2}{m(s)} \right). \end{aligned} \quad (\text{A.7})$$

The first order conditions for consumption and investment in (12) and (13) then become:

$$\zeta U'(c) = b^{1-\gamma} m(s) m'(s)^{-\gamma}, \quad (\text{A.8})$$

$$1 + g'(i) = \frac{m(s)}{m'(s)} - s. \quad (\text{A.9})$$

From the first order conditions (14) and (15), we obtain (39) and (40).

Finally, substituting these policy functions for $c(s)$, $i(s)$, $\phi_1(s)$ and $\phi_2(s)$ into (A.7), we

obtain the ODE given in (49) for $m(s)$:

$$0 = \frac{m(s)}{1-\gamma} \left[\gamma \chi m'(s)^{\frac{\gamma-1}{\gamma}} - \zeta \right] + [rs + A - i(s) - g(i(s))] m'(s) + (i(s) - \delta)(m(s) - sm'(s)) - \left(\frac{\gamma \sigma_K^2}{2} - \rho \eta \sigma_K \right) \frac{m(s)^2 m''(s)}{m(s) m''(s) - \gamma m'(s)^2} + \frac{\eta^2 m'(s)^2 m(s)}{2(\gamma m'(s)^2 - m(s) m''(s))}, \quad (\text{A.10})$$

where χ is defined by

$$\chi \equiv b^{\frac{\gamma-1}{\gamma}} \zeta^{\frac{1}{\gamma}}. \quad (\text{A.11})$$

A.1 First Best

Under the first-best case, we have $m^{FB}(s) = s + q^{FB}$. Substituting for $m^{FB}(s)$ into the ODE (A.10) we obtain:

$$0 = \frac{s + q^{FB}}{1-\gamma} [\gamma \chi - \zeta] + [rs + A - i^{FB} - g(i^{FB})] + (i^{FB} - \delta) q^{FB} + \frac{\eta^2 (s + q^{FB})}{2\gamma} = \left(\frac{\gamma \chi - \zeta}{1-\gamma} + \frac{\eta^2}{2\gamma} + r \right) (s + q^{FB}) + [A - i^{FB} - g(i^{FB}) - (r + \delta - i^{FB}) q^{FB}]. \quad (\text{A.12})$$

As (A.12) must hold for all $m^{FB}(s) = s + q^{FB}$, we must have

$$\chi = r + \frac{\eta^2}{2\gamma} + \gamma^{-1} \left(\zeta - r - \frac{\eta^2}{2\gamma} \right), \quad (\text{A.13})$$

as given by (26), and

$$0 = A - i^{FB} - g(i^{FB}) - (r + \delta - i^{FB}) q^{FB}, \quad (\text{A.14})$$

so that (19) holds. Finally, using (A.11), we obtain the expression (17) for the coefficient b .

Next, substituting $m(s) = m^{FB}(s) = s + q^{FB}$ into (A.8) and (A.9) gives the first-best consumption rule (25) and investment policy (21). Moreover, substituting $m(s) = m^{FB}(s) = s + q^{FB}$ into ((39) and (40) respectively, we obtain the first-best idiosyncratic risk hedge $\phi_1^{FB}(s)$ given in (27) and the market portfolio allocation $\phi_2^{FB}(s)$ given in (28).

Turning to the dynamics of $\{M_t^{FB}; t \geq 0\}$, we apply the Ito's formula to $M_t^{FB} = S_t +$

$Q_t^{FB} = S_t + q^{FB}K_t$ and obtain the following dynamics:

$$\begin{aligned}
dM_t^{FB} &= d(S_t + Q_t^{FB}) = (rS_t + Y_t - C_t)dt + \Phi_{1,t}^{FB}\epsilon_K d\mathcal{Z}_{1,t} + \Phi_{2,t}^{FB}[(\mu_2 - r)dt + \sigma_2 d\mathcal{Z}_{2,t}] \\
&\quad + Q_t^{FB} [(i^{FB} - \delta_K) dt + (\epsilon_K d\mathcal{Z}_{1,t} + \rho\sigma_K d\mathcal{Z}_{2,t})] \\
&= M_t^{FB} \left[\left(r - \chi + \frac{\eta^2}{\gamma} \right) dt + \frac{\eta}{\gamma} d\mathcal{Z}_{2,t} \right], \tag{A.15}
\end{aligned}$$

where we use the following first-best policy rules:

$$\Phi_{1,t}^{FB}(s) = -Q_t^{FB} \quad \text{and} \quad \Phi_{2,t}^{FB}(s) = -\beta^{FB}Q_t^{FB} + \frac{\eta}{\gamma\sigma_2} (S_t + Q_t^{FB}) .$$

A.2 Limited Commitment

First, recall that for the limited-commitment case, the HJB equation is the same as that for the first-best case. We thus only focus on boundary conditions here. At the left boundary $\underline{S}(K)$, the entrepreneur's credit constraint binds, which implies:

$$J(K, \underline{S}) = J(\alpha K, 0) . \tag{A.16}$$

By substituting the value function (35) into (A.16), we obtain $M(K, \underline{S}) = M(\alpha K, 0)$, which implies (45). The boundary conditions given in (46) are necessary to ensure that the entrepreneur will stay with the firm, which implies that

$$\phi_1(\underline{s}) = \underline{s}, \quad \phi_2(\underline{s}) = \underline{s}\beta^{FB} . \tag{A.17}$$

And then comparing the above equations with (39) and (40), it is straightforward to show that (46) is equivalent to $\lim_{s \rightarrow \underline{s}} m''(s) = -\infty$ as given in (47). Finally, we show that as $s \rightarrow \infty$, we have $\lim_{s \rightarrow \infty} m(s) = m^{FB}(s) = s + q^{FB}$.

For brevity, we omit standard arguments as in Krylov (1980) and Karatzas and Shreve (1991) to establishing existence and uniqueness of the solution for the optimization problem.

B Optimal Contract

B.1 Solution of the Contracting Problem

In this subsection, we provide the derivations for Propositions 5 and 6.

HJB Equation. First, we recall that the investors' optimization problem is

$$F(K_t, V_t) = \max_{C, I, x_1, x_2} \mathbb{E}_t \left[\int_t^\infty \frac{\mathbb{M}_v}{\mathbb{M}_t} (Y_v - C_v) dv \right], \quad (\text{B.1})$$

subject to the entrepreneurs' inalienability-of-human-capital constraints (51) for all time t , and the entrepreneur's initial participation constraint (54).

Using Ito's formula, we have

$$d(\mathbb{M}_t F(K_t, V_t)) = \mathbb{M}_t dF(K_t, V_t) + F(K_t, V_t) d\mathbb{M}_t + \langle d\mathbb{M}_t, dF(K_t, V_t) \rangle, \quad (\text{B.2})$$

where

$$\begin{aligned} dF(K_t, V_t) &= F_K dK_t + \frac{F_{KK}}{2} \langle dK_t, dK_t \rangle + F_V dV_t \\ &\quad + \frac{F_{VV}}{2} \langle dV_t, dV_t \rangle + F_{VK} \langle dV_t, dK_t \rangle \\ &= \left[(I - \delta_K K) F_K + \frac{\sigma_K^2 K^2 F_{KK}}{2} + \zeta (V - U(C)) F_V \right] dt \\ &\quad + \left[\frac{(x_1^2 + x_2^2) V^2 F_{VV}}{2} + (x_1 \epsilon_K + x_2 \rho \sigma_K) K V F_{VK} \right] dt \\ &\quad + V F_V (x_1 dZ_{1,t} + x_2 dZ_{2,t}) + \sigma_K K F_K \left(\sqrt{1 - \rho^2} dZ_{1,t} + \rho dZ_{2,t} \right). \end{aligned} \quad (\text{B.3})$$

Using the SDF \mathbb{M} given in (9) and the following martingale representation,

$$\mathbb{E}_t[d(\mathbb{M}_t F(K_t, V_t))] + \mathbb{M}_t (Y_t - C_t) dt = 0, \quad (\text{B.4})$$

we obtain (57), which is the HJB equation for the optimal contracting problem.

First-Order Conditions. Differentiating the right-hand side of (57) with respect to C , I , x_1 , and x_2 we then obtain the following FOCs:

$$\zeta U'(C^*) = -\frac{1}{F_V(K, V)}, \quad (\text{B.5})$$

$$F_K(K, V) = 1 + G_I(I^*, K), \quad (\text{B.6})$$

$$x_1^* = -\frac{\epsilon_K K F_{VK}}{V F_{VV}(K, V)}, \quad (\text{B.7})$$

$$x_2^* = -\frac{\rho \sigma_K K F_{VK}}{V F_{VV}(K, V)} + \frac{\eta F_V}{V F_{VV}(K, V)}. \quad (\text{B.8})$$

FOC (B.5) equates the entrepreneur's marginal utility of consumption $\zeta U'(C^*)$ with $-1/F_V$, which is positive as $F_V < 0$. Multiplying (B.5) through by $-F_V$, we observe that at the optimum the agent's normalized marginal utility of consumption, $-F_V \zeta U'(C)$, has to equal unity, the risk-neutral investor's marginal cost of providing a unit of consumption. Note that (B.5) is analogous to the inverse Euler equation in Rogerson (1985).

FOC (B.6) characterizes optimal investment, which is obtained when the marginal benefit of investing, $F_K(K, V)$, is equal to the marginal cost of investing, $1 + G_I(I, K)$. FOC (B.7) and (B.8) characterize the optimal exposures of the promised utility V to the idiosyncratic shock \mathcal{Z}_1 and the systematic shock \mathcal{Z}_2 , respectively. As we show later, x_1 and x_2 are closely tied to the firm's optimal risk management policies $\phi_1(s)$ and $\phi_2(s)$, respectively.

Dynamics of the Entrepreneur's Promised Scaled Wealth w . Using Ito's lemma, we have the following dynamics for W :

$$dW_t = \frac{\partial W}{\partial V} dV_t + \frac{1}{2} \frac{\partial^2 W}{\partial V^2} \langle dV_t, dV_t \rangle = \frac{dV_t}{V_W} - \frac{V_{WW}}{2V_W^3} \langle dV_t, dV_t \rangle, \quad (\text{B.9})$$

where $\langle dV_t, dV_t \rangle$ denotes the quadratic variation of V , (B.9) uses $\partial W / \partial V = 1/V_W$, and

$$\frac{\partial^2 W}{\partial V^2} = \frac{\partial V_W^{-1}}{\partial V} = \frac{\partial V_W^{-1}}{\partial W} \frac{\partial W}{\partial V} = -\frac{V_{WW}}{V_W^2} \frac{1}{V_W} = -\frac{V_{WW}}{V_W^3}. \quad (\text{B.10})$$

Substituting the dynamics of V given by (56) into (B.9) yields

$$dW_t = \frac{1}{V_W} [\zeta(V - U(C_t))dt + x_1 V d\mathcal{Z}_{1,t} + x_2 V_t d\mathcal{Z}_{2,t}] - \frac{(x_1^2 + x_2^2)V^2 V_{WW}}{2V_W^3} dt. \quad (\text{B.11})$$

Using the dynamics for W and K , we can write the dynamic evolution of the certainty equivalent wealth w as follows:

$$dw_t = d\left(\frac{W_t}{K_t}\right) = \mu^w(w)dt + \sigma_1^w(w)d\mathcal{Z}_{1,t} + \sigma_2^w(w)d\mathcal{Z}_{2,t}, \quad (\text{B.12})$$

where the drift and volatility processes $\mu^w(\cdot)$ and $\sigma_1^w(\cdot)$ and $\sigma_2^w(\cdot)$ for w are given by

$$\mu^w(w) = \frac{\zeta}{1-\gamma} \left(w + \frac{\mathbf{c}(w)}{\zeta p'(w)} \right) - w(i(w) - \delta_K) + \frac{\gamma w}{2} \frac{x_1^2 + x_2^2}{(1-\gamma)^2} - (\epsilon_K \sigma_1^w(w) + \rho \sigma_K \sigma_2^w(w)), \quad (\text{B.13})$$

and

$$\sigma_1^w(w) = -w \left(\epsilon_K - \frac{x_1(w)}{1-\gamma} \right), \quad \sigma_2^w(w) = -w \left(\rho\sigma_K - \frac{x_2(w)}{1-\gamma} \right). \quad (\text{B.14})$$

Derivation for Propositions 5 and 6. Applying Ito's formula to (58) and transforming (57) for $F(K, V)$ into an HJB equation for $P(K, W)$, we obtain the following:

$$\begin{aligned} rP(K, W) = \max_{C, I, x_1, x_2} \left\{ & Y - C + \frac{\zeta(U(bW) - U(C)) - x_2\eta U(bW)}{bU'(bW)} P_W \right. \\ & + (I - \delta_K K - \rho\eta\sigma_K K) P_K + \frac{\sigma_K^2 K^2}{2} P_{KK} \\ & + \frac{(x_1^2 + x_2^2)(U(bW))^2}{2} \frac{P_{WW} bU'(bW) - P_W b^2 U''(bW)}{(bU'(bW))^3} \\ & \left. + (x_1\epsilon_K + x_2\rho\sigma_K) \frac{KU(bW)}{bU'(bW)} P_{WK} \right\}. \end{aligned} \quad (\text{B.15})$$

And then using the FOCs for I , C , x_1 and x_2 respectively, we obtain

$$1 + G_I(I, K) = P_K(K, W), \quad (\text{B.16})$$

$$U'(bW) = -\frac{\zeta}{b} P_W(K, W) U'(C), \quad (\text{B.17})$$

$$x_1 = -\frac{\epsilon_K K P_{WK} bU'(bW)}{U(bW)[P_{WW} - P_W bU''(bW)/U'(bW)]}, \quad (\text{B.18})$$

$$\begin{aligned} x_2 = & -\frac{\rho\sigma_K K P_{WK} bU'(bW)}{U(bW)[P_{WW} - P_W bU''(bW)/U'(bW)]} \\ & + \frac{\eta P_W bU'(bW)}{U(bW)[P_{WW} - P_W bU''(bW)/U'(bW)]}. \end{aligned} \quad (\text{B.19})$$

Substituting $P(K, W) = p(w)K$ into (B.16)-(B.19), we obtain the optimal investment, consumption, and risk management policies given by (63)-(66), respectively. And then substituting $P(K, W) = p(w)K$ and these optimal policies into the PDE (B.15), we obtain the ODE (59). The constraint $V_t \geq \widehat{V}(K_t)$ implies $W_t \geq \underline{w}K_t$ where \underline{w} is denoted by $\underline{w} = U^{-1}(\widehat{V}(K_t))/(bK_t)$. To ensure $V_{t+dt} \geq \widehat{V}(K_{t+dt})$ with probability one, the drift of $V_t/\widehat{V}(K_t)$ should be weakly positive (negative) if $V_t > 0$ ($V_t < 0$) and the volatility of $V_t/\widehat{V}(K_t)$ should be zero at the boundary $V_t = \widehat{V}(K_t)$. Therefore, the following conditions ensure that the entrepreneur will not walk away at the endogenous left boundary \underline{w} :

$$\lim_{w \rightarrow \underline{w}} \sigma_1^w(w) = 0, \quad \lim_{w \rightarrow \underline{w}} \sigma_2^w(w) = 0, \quad \text{and} \quad \lim_{w \rightarrow \underline{w}} \mu^w(w) \geq 0. \quad (\text{B.20})$$

Using (B.14), (65), and (66), we show that the conditions given in (B.20) are equivalent to the boundary condition $\lim_{w \rightarrow \underline{w}} p''(w) = -\infty$. We determine \underline{w} by using (52) and (58), which together imply

$$P(\alpha K, \overline{W}(\alpha K)) = 0. \quad (\text{B.21})$$

We thus have $p(\overline{w}) = 0$, where $\overline{w} = \overline{W}(\alpha K)/(\alpha K)$ and $\overline{W}(\alpha K) = U^{-1}(\overline{V}(\alpha K))/b$. In addition, (53) implies $\underline{w} = \alpha \overline{w}$, where $\underline{w} = \underline{W}(K)/K$ and $\underline{W}(K) = U^{-1}(\widehat{V}(K))/b$. Furthermore, substituting $\underline{w} = \alpha \overline{w}$ into $p(\overline{w}) = 0$, we obtain $p(\underline{w}/\alpha) = 0$ as given in (61). Finally, the boundary condition (62) ensures that $p(w)$ attains the first-best as $w \rightarrow \infty$.

The Two-Sided Limited-Commitment Case. When investors face the limited-liability constraint, the contract requires the volatility at the endogenous upper boundary \overline{w} to be zero and additionally the drift to be non-positive in order for the investors not to walk away from the contractual agreement:

$$\lim_{w \rightarrow \overline{w}} \sigma_1^w(w) = 0, \quad \lim_{w \rightarrow \overline{w}} \sigma_2^w(w) = 0, \quad \text{and} \quad \lim_{w \rightarrow \overline{w}} \mu^w(w) \leq 0. \quad (\text{B.22})$$

The arguments for (B.22) are essentially the same as those we have sketched out for the lower boundary $\underline{w} = \alpha \overline{w}$. In addition, the boundary condition (B.22) implies the equivalent boundary condition:

$$\lim_{w \rightarrow \overline{w}} p''(w) = -\infty. \quad (\text{B.23})$$

Again, for brevity, we omit standard arguments as in Krylov (1980) and Karatzas and Shreve (1991) to establishing existence and uniqueness of the solution for the entrepreneur's optimization problem.

B.2 Equivalence between the Contracting and Dual Problems

Having characterized the optimal contract in terms of the entrepreneur's promised certainty-equivalent wealth W , we show next how to implement the optimal contract by flipping the optimal contracting problem on its head and considering a dynamic entrepreneurial finance problem, where the entrepreneur owns the firm's productive, illiquid capital stock and chooses consumption and corporate investment by optimally managing liquidity and risk subject only to satisfying the endogenous liquidity constraint. A key observation is that the entrepreneur's inalienability-of-human-capital constraints naturally translate to endogenous liquidity constraints in the entrepreneur's problem.

This dual optimization problem for entrepreneur is equivalent to the optimal contract

problem for the investor in (50) if and only if the borrowing limits, $\underline{S}(K)$, are such that:

$$\underline{S}(K) = -P(K, \underline{W}), \quad (\text{B.24})$$

where $P(K, \underline{W})$ is the investors' value when the entrepreneur's inalienability-of-human-capital constraint binds, that is, when $W = \underline{W}$. Accordingly, we characterize the implementation solution for the dual problem by first solving the investors' problem in (57), and then imposing the constraint in (B.24).

To summarize, the primal optimal contracting problem gives rise to the investor's value function $F(K, V)$, with the promised utility to the entrepreneur V as the key state variable. By expressing V in units of consumption rather than utils, the investor's value $F(K, V)$ can be expressed in terms of the entrepreneur's promised certainty-equivalent wealth W : $P(K, W)$. The dual problem for the entrepreneur gives rise to the entrepreneur's value function $J(K, S)$, with $S = -P(K, W)$ as the key state variable. Or, again expressing the entrepreneur's value in units of consumption, the entrepreneur's value function is her certainty equivalent wealth $M(K, S)$ and the relevant state variable is her savings $S = -P$.

Next, we provide a proof for the equivalence between the contracting and dual problems, as Table 1 in Introduction summarizes. First, we show that the following relations between s and w hold:

$$s = -p(w) \text{ and } m(s) = w. \quad (\text{B.25})$$

Then, the standard chain rule implies:

$$m'(s) = -\frac{1}{p'(w)} \text{ and } m''(s) = -\frac{p''(w)}{p'(w)^3}. \quad (\text{B.26})$$

Substituting (B.25) and (B.26) into the ODE (49) for $m(s)$, we obtain the ODE (59) for $p(w)$. Similarly, substituting (B.25) and (B.26) into (46) for $m(s)$, we obtain (62), a boundary condition for $p(w)$. Substituting (B.25) into (48) for $m(s)$, we obtain $\lim_{s \rightarrow \infty} p(w) = q^{FB} - w$. And then substituting (B.25) into (45), we obtain (61). Finally, by using (B.25) and (B.26), we show that the optimal consumption and investment policies for the primal and dual problems are indeed equivalent.

C Persistent Productivity Shocks

First, by using the dynamics of S_t given by (75) and the dynamic programming method, we obtain the HJB equation for the entrepreneur's value function $J^L(K, S)$ in State L as given

in (76) and similarly the following HJB equation for $J^H(K, S)$ in State H :

$$\begin{aligned}
\zeta J^H(K, S) = & \max_{C, I, \Phi_1, \Phi_2, \Pi^H} \zeta U(C) + (I - \delta_K K) J_K^H + \frac{\sigma_K^2 K^2}{2} J_{KK}^H \\
& + (rS + \Phi_2(\mu_2 - r) + A^H K - I - G(I, K) - C - \lambda^H \Pi^H) J_S^H \\
& + (\epsilon_K^2 \Phi_1 + \rho \sigma_K \sigma_2 \Phi_2) K J_{KS}^H + \frac{(\epsilon_K \Phi_1)^2 + (\sigma_2 \Phi_2)^2}{2} J_{SS}^H \\
& + \lambda^H [J^L(K, S + \Pi^H) - J^H(K, S)]. \tag{C.1}
\end{aligned}$$

We then obtain the following main results:

Proposition 7 *In the region $s > \underline{s}^L$, $m^L(s)$ satisfies the following ODE:*

$$\begin{aligned}
0 = & \max_{i^L, \pi^L} \frac{m^L(s)}{1 - \gamma} \left[\gamma \chi m^{L'}(s)^{\frac{\gamma-1}{\gamma}} - \zeta \right] + [rs + A^L - i^L - g(i^L) - \lambda^L \pi^L(s)] m^{L'}(s) \\
& - \left(\frac{\gamma \sigma_K^2}{2} - \rho \eta \sigma_K \right) \frac{m^L(s)^2 m^{L''}(s)}{m^L(s) m^{L''}(s) - \gamma m^{L'}(s)^2} + \frac{\eta^2 m^{L'}(s)^2 m^L(s)}{2(\gamma m^{L'}(s)^2 - m^L(s) m^{L''}(s))} \\
& + (i^L - \delta)(m^L(s) - s m^{L'}(s)) + \frac{\lambda^L m^L(s)}{1 - \gamma} \left(\left(\frac{m^H(s + \pi^L)}{m^L(s)} \right)^{1-\gamma} - 1 \right), \tag{C.2}
\end{aligned}$$

subject to the following boundary conditions:

$$\lim_{s \rightarrow \infty} m^L(s) = q_L^{FB} + s, \tag{C.3}$$

$$m^L(\underline{s}^L) = \alpha m^L(0), \tag{C.4}$$

$$\lim_{s \rightarrow \underline{s}^L} m^{L''}(s) = -\infty. \tag{C.5}$$

where q_L^{FB} is provided in Proposition 8. Finally, the (scaled) insurance demand $\pi^L(s)$ solves the following implicit equation:

$$m^{H'}(s + \pi^L) = m^{L'}(s) \left(\frac{m^L(s)}{m^H(s + \pi^L)} \right)^{-\gamma}, \tag{C.6}$$

provided that the solution $\pi^L(s)$ to the above FOC also satisfies $\pi^L(s) \geq \underline{s}^H - s$. Otherwise, the entrepreneur sets $\pi^L(s)$ as follows:

$$\pi^L = \underline{s}^H - s. \tag{C.7}$$

We have another set of essentially the same equations and boundary conditions for $m^H(s)$

and $\pi^H(s)$ in state H .

The following proposition summarizes the solutions for the first-best case.

Proposition 8 *Under the first-best, the firm's value $Q_n^{FB}(K)$ in state $n = \{H, L\}$ is proportional to K : $Q_n^{FB}(K) = q_n^{FB}K$, where q_H^{FB} and q_L^{FB} jointly solve:*

$$(r + \delta - i_L^{FB}) q_L^{FB} = A^L - i_L^{FB} - g(i_L^{FB}) + \lambda^L (q_H^{FB} - q_L^{FB}), \quad (\text{C.8})$$

$$(r + \delta - i_H^{FB}) q_H^{FB} = A^H - i_H^{FB} - g(i_H^{FB}) + \lambda^H (q_L^{FB} - q_H^{FB}), \quad (\text{C.9})$$

and where i_L^{FB} and i_H^{FB} satisfy:

$$q_L^{FB} = 1 + g'(i_L^{FB}) \quad \text{and} \quad q_H^{FB} = 1 + g'(i_H^{FB}). \quad (\text{C.10})$$

The insurance demands in state L and H are respectively given by:

$$\pi^L = q_H^{FB} - q_L^{FB} \quad \text{and} \quad \pi^H = q_L^{FB} - q_H^{FB}. \quad (\text{C.11})$$